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An approximate renormalization for the break-up of invariant tori with three frequencies

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Abstract

Renormalization theory provides a description of the destruction of invariant tori for Hamiltonian systems of $1\frac{1}{2}$ or 2 degrees of freedom, and explains the self-similarity and the universality of the structures observed A similar theory for higher dimensional Hamiltonian systems has proved elusive Here we construct an approximate renormalization for a Hamiltonian system with $2\frac{1}{2}$ degrees of freedom analogous to the lower dimensional version of Escande and Doveil. Using this operator we study the critical surface for the "spiral mean" invariant torus We find that there is no universal fixed point Instead the renormalization dynamics on the critical surface is a rotation with irrational winding ratio. Implications for the determination of the exact critical surface are discussed.

1. Introduction

The Kolmogorov–Arnol'd–Moser (KAM) theorem implies that sufficiently incommensurate invariant tori of integrable Hamiltonian systems or symplectic maps are preserved for small enough perturbations. Alternatively, in many such systems one can use "converse-KAM" theory to show that for certain parameter and frequency ranges there will be no invariant tori continuously deformable to those of the integrable case [1–4]. The set of parameters for which there exists an invariant torus of a given Diophantine frequency with smooth conjugacy to rotation and the set for which there does not exist any invariant torus of that frequency are both open. The destruction of invariant tori signals the onset of chaos, and (particularly for two degrees of freedom) the loss of stability. Thus it is of great interest to develop techniques for studying this destruction.

In 1979 Greene [5] made the remarkable discovery in embryonic form that the phase space of an area preserving map exhibits self-similarity in the neighborhood of a critical invariant circle with golden mean winding ratio. This observation led to the construction of renormalization operator on the space of area preserving maps [6,7]. This operator has also been extended to general winding ratios [8,9]

There have been many unsuccessful attempts to find a similar self-similarity for the breakup of tori in higher dimensions [10–15]. In these cases the authors studied three frequency systems: either maps of the torus, volume preserving maps, or four dimensional symplectic maps.

In order to discover the reason for this failure, we will construct an analytic approximation to the re-

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normalization operator for $2\frac{1}{2}$ degrees of freedom. Our construction is analogous to that of Escande and Doveil [16,17] who considered the case of $1\frac{1}{2}$ degrees of freedom. Their approximate renormalization provides a simple model that has many of the features of the exact renormalization and yields, within a few percent, the main eigenvalues near the critical golden fixed point. It can also be used to discuss such issues as the robustness of tori with different rotation numbers and the expected rotation numbers for boundary tori [18-20] - though it loses accuracy for frequencies that have large continued fraction elements.

Escande and Doveil studied a model corresponding to the motion of a particle in one dimension acted on by two waves with different wavenumbers (without loss of generality 1 and k) and phase velocities (without loss of generality 0 and 1):

$$H(p, x, t)=\tfrac{1}{2}p^2+A\cos(2\pi x)+B\cos[2\pi k(x-t)]$$

This Hamiltonian has three parameters and is periodic of period 1 in the configuration variables (x, $y=k(x-t)$). When $A=B=0$ there is a torus for each p with frequency vector $\omega=(\dot{x}, \dot{y})$. The renormalization is a map $\mathscr{R}(A, B, k)=(A', B', k')$ of the parameters giving a modified Hamiltonian describing the system in new coordinates. These are essentially obtained by a shear in the configuration variables and a magnification of the momentum. The shear is arranged so that the new configuration coordinates are more closely aligned with the frequency vector of the invariant torus of interest. The shear is a unimodular transformation (to maintain periodicity) and it corresponds to one step of the Farey expansion for ω The magnification focuses on a layer around the position of the torus. The simplest way to obtain such a map is to perform the canonical and rescaling transformations to lowest order in the amplitudes A and B, though one can be much more sophisticated. The Hamiltonian in the new coordinate system takes a form identical to the original one upon mapping the parameters.

We will construct a similar operator for a system of $2\frac{1}{2}$ degrees of freedom. We also choose a Hamiltonian corresponding to a particle in a time dependent potential. One important new feature is the presence of a mass matrix in the kinetic energy, we will see that the off-diagonal elements of this matrix are essential The second new feature is the use of the Kim-Ostlund generalization of the Farey tree [21] to construct the shear transformation.

We begin by defining the model Hamiltonian.

2. Model Hamiltonian

A particle in the plane that is acted on by a potential from three electrostatic waves has the Hamiltonian

$$H=\frac{1}{2m}\boldsymbol{p}^2+\sum_{i=1}^{3}\varphi_i\cos(\boldsymbol{k}_i\cdot\boldsymbol{x}-\omega_i t)$$

Providing that none of the wavevectors $\boldsymbol{k}_i$ are parallel, we can choose new canonical coordinates (x, u) and (y, v) to transform H to the standard form

$$H=\tfrac{1}{2}(u, v)\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\cdot\begin{pmatrix}u\\ v\end{pmatrix}+V(x, ky, lz),$$

$$V=A\cos(2\pi x)+B\cos(2\pi ky)+C\cos(2\pi lz),$$

$$z=t-x-y. \quad (1)$$

Without loss of generality, the wavenumbers (k, l) can be taken to be positive and the energy can be scaled so that the mass matrix has unit determinant, $\alpha\gamma-\beta^2=1$. Thus we obtain a seven parameter system.

It is useful to think of the motion on the five dimensional extended phase space with coordinates $\zeta=(x, y, z, u, v)$. The Hamiltonian is periodic with periods $(1, 1/k, 1/l)$ in the configuration variables (x, y, z) - so the configuration space can be taken to be the three torus $T^3=\{x \bmod 1, y \bmod 1/k, z \bmod 1/l\}$. Note that one could incorporate the factors k and l into the definitions of y and z to make the periods unity, but we did not do so. The frequency vector ω is the average direction that an orbit moves around the torus

$$\omega=\lim_{\Delta t\to\infty}\frac{1}{\Delta t}(\Delta x, k\Delta y, l\Delta z)$$

if the limit exists. We care only about its direction (defined by two winding ratios), so ω should be viewed as a point in the projective space RP^2. The frequency is *commensurate* if there is a nonzero integer vector m such that $m\,\omega=0$. Such a relation is a *resonance* condition. If there are no resonances for ω then it is *incommensurate*. If there are two independent resonances for ω, then $\omega=p$ where p is integral (remember the length of ω is unimportant). A frequency ω is *Diophantine* if there is a $K\neq 0$ and $\tau>2$ such that $\forall\ m\in Z^3\setminus 0$, $|m\cdot\omega|/|\omega|>K/|m|^\tau$.

When $A=B=C=0$, the momenta (u, v) are constant in time and every orbit lies on a three torus. If $\omega(u, v)$ is incommensurate, the orbit densely covers the torus. If ω is Diophantine, then the KAM theorem implies that there is a torus with this frequency for small values of the amplitudes. We are interested in determining the parameters for which such a torus is destroyed.

The technique is to perform a succession of canonical transformations to coordinates that are more closely aligned with the incommensurate flow. We use the Kim–Ostlund extension of the Farey algorithm to successively construct these coordinates.

3. Kim–Ostlund tree

Each of the three phases in $V(x, y, z)$ corresponds to a resonance $m_i\cdot\omega=0$. We begin with the three resonances $m_1=(1,0,0)$, $m_2=(0,1,0)$, $m_3=(0,0,1)$. Each resonance corresponds to a plane in R^3 or a line in RP^2; the set of three resonances delineates a cone (the positive octant) or triangle, see Fig. 1. The intersection of each pair of resonances defines rational frequencies $p_1=[1, 0, 0]$, $p_2=[0, 1, 0]$, $p_3=[0, 0, 1]$ The frequencies p_i also delineate the cone; it is the convex hull of the three vectors. We denote the cone by either of the matrices

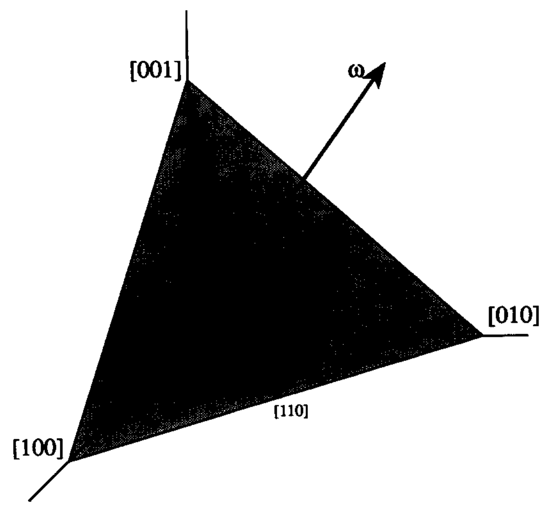


Fig 1 Farey triangle construction Frequency ratios are denoted by [ ], and resonances by ( ) The frequency shown has the Farey sequence $\omega=LL$

$$M=\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad P=(p_1, p_2, p_3)\,.$$

We assume ω is inside the cone, i.e. $\omega_i\geqslant 0$.

To construct the Farey sequence for ω, divide the cone using the new frequency $p'=p_1+p_2$, and corresponding resonance $m'=m_1-m_2$. There is now a right and a left cone $P_R=(p_3, p_1, p')$ and $P_L=(p_2, p_3, p')$, or $M_R=(m_3, m', m_2)^t$ and $M_L=(-m', m_3, m_1)^t$. Choose the new cone that contains ω and repeat this transformation, dividing this new cone into two. This gives a sequence of cones that each contain ω. The operations can be represented by the linear transformations

$$M_S=S^{-1}M, \quad P_S=PS\,,$$

$$S=R \quad \text{if } (m_1-m_2)\cdot\omega>0\,,$$

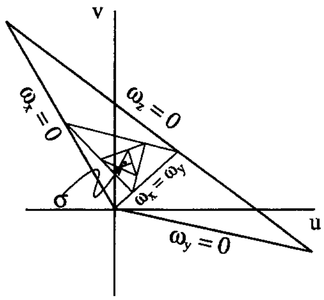
$$=L \quad \text{if } (m_1-m_2)\cdot\omega\leqslant 0\,,$$

$$R=\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad L=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2}$$

Note that $\det(R)=\det(L)=1$, $M_SP_S=I$, and $\det(M_S)=\det(P_S)=1$. Repeating this transformation provides a unique string of matrices $S_i\in\{R, L\}$ for any ω, so that we can think of ω as the sequence $S_1S_2S_3\ldots$. It is not difficult to show that if ω is an integer vector (with no common factors) then this sequence eventually terminates when $p_3=\omega$ [22].

From the Farey point of view, the simplest incommensurate frequency vectors have periodic Farey sequences. When the period is q, ω is the eigenvector with the largest eigenvalue of the nonnegative matrix $S_1\ldots S_q$. This implies that the components of ω are elements of a cubic field: they satisfy $\omega_i=i+j\lambda+k\lambda^2$, where (i, j, k) are integers and λ is the eigenvalue – it satisfies a cubic equation with integer coefficients. The simplest of these is the spiral mean, which is the eigenvector of L:

$$\omega=(1, \sigma^2, \sigma), \quad \sigma^3=\sigma+1, \quad \sigma\approx 1.324717957\,. \tag{3}$$



Similarly the eigenvector of R is $(\sigma^2, 1, \sigma)$

For the model (1) with $A=B=C=0$, the frequency vector for momenta (u, v) is

$$\omega = (\alpha u + \beta v,\ k(\beta u + \gamma v),$$
$$l[1-(\alpha+\beta)u-(\beta+\gamma)v]),$$

and each resonance $m_i\ \omega=0$ corresponds to a line in momentum space, as shown in Fig 2. Tori with rational frequency ratio, hence consisting entirely of periodic orbits, occur at the intersections of the resonance lines.

4. Renormalization transformation

Our renormalization is a coordinate transformation that focuses in on a region of phase space in which orbits of a given frequency ratio are expected. Here we will define two such transformations corresponding to the L and R Farey steps. We define a canonical transformation to eliminate one of the resonances and then transform the new Hamiltonian back to its original form

Suppose formally that each of the parameters A, B, $C=O(\epsilon)$. We begin by eliminating the $m_2=(010)$ resonance by a near identity canonical transformation as shown in the Appendix. Then, for the "L" transformation, define the new coordinates on T^3

$$\begin{pmatrix} x' \\ k'y' \\ l'z' \end{pmatrix} = L^{-1} \begin{pmatrix} x \\ ky \\ lz \end{pmatrix} + O(\epsilon) \quad (4)$$

In order to maintain the form $z'=t'-x'-y'$ the new wavenumbers must be

$$\mathscr{L}:\ k'=l/k,\quad l'=1/(1+k) \quad (5)$$

Upon defining new momenta corresponding to these coordinates, scaling time to $t'=kt$ and scaling the momenta to normalize the mass matrix, the Hamiltonian has the same form as (1) to $O(\epsilon^3)$ if we identify the new parameters

$$\mathscr{L}\quad \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = \frac{1}{1+k} \begin{pmatrix} 1/k & -2 & k \\ 1 & 1-k & -k \\ k & 2k & k \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix},$$
$$A' = \frac{(1+k)^3\beta}{2k^2} AB,\quad B' = \frac{1+k}{k} C,$$
$$C' = \frac{1+k}{k} A \quad (6)$$

This is the approximate renormalization operator. The transformation corresponding to R is conjugate to $\mathscr{L}$, $\mathscr{R} = \mathscr{F}\mathscr{L}\mathscr{F}$ under the involution

$$\mathscr{F}\quad (k, l, \alpha, \beta, \gamma, A, B, C)$$
$$\to (1/k, l/k, \gamma, \beta, \alpha, B, A, C)\,. \quad (7)$$

5. Kinematic renormalization

The wavenumber and mass renormalization maps are independent of the amplitude They arise from the purely kinematical rescaling transformations. Consider these first.

The simplest frequency vectors under renormalization correspond to the fixed points of the $\mathscr{R}$ and $\mathscr{L}$ operators. As they are essentially identical, we consider the $\mathscr{L}$ fixed point here, the frequency vector is then the spiral mean (3).

The wavenumber renormalization (5) is decoupled from the parameters, so we can consider it separately The wavenumber map takes the positive quadrant to the strip $\{0<k,\ 0<l\leqslant 1\}$, which is mapped onto itself. Furthermore $\mathscr{L}^5$ maps the positive quadrant into the triangle $\{\frac{1}{2}\leqslant l\leqslant k\leqslant 1\}$, so the map is contracting. Thus there is a unique real fixed point

$$l=k^2=\frac{1}{1+k} \quad\Rightarrow\quad k=\sigma^{-1},\quad l=\sigma^{-2}, \quad (8)$$

where σ is the spiral mean. The fixed point is a spiral focus with linearization

$$\begin{pmatrix}\delta k'\\ \delta l'\end{pmatrix}=\begin{pmatrix}-1 & \sigma\\ -\sigma^{-4} & 0\end{pmatrix}\begin{pmatrix}\delta k\\ \delta l\end{pmatrix}.$$

The eigenvalues are

$$\lambda=\sigma^{-3/2}e^{\pm i\psi/2},$$

$$\cos(\psi)=\tfrac{1}{2}(\sigma-1),\quad \psi\approx 2\pi\times 0.22404487. \tag{9}$$

The mass renormalization is a linear map. Recall that it has been constructed to preserve the subspace $\alpha\gamma-\beta^2=1$. Since the wavenumber map is contracting, we can evaluate the mass map at the fixed point $k=\sigma^{-1}$. This gives the eigenvalues

$$\lambda_1=1,\quad \lambda_{2,3}=e^{\pm i\psi}. \tag{10}$$

Therefore this map is not contracting – in general the mass matrix rotates with a rotation number of $\psi/2\pi$ that is very nearly $\frac{2}{9}$. The eigenvectors are

$$\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=\begin{pmatrix}\csc(\psi)\\ \cot(\psi)\\ \csc(\psi)\end{pmatrix},\begin{pmatrix}\cos(\psi)\\ 1\\ \cos(\psi)\end{pmatrix}\pm i\begin{pmatrix}\sin(\psi)\\ 0\\ -\sin(\psi)\end{pmatrix}.$$

The first eigenvector corresponds to a fixed point, but the general orbit of the mass matrix is a rotation about this fixed point. The general orbit of β under $\mathscr{L}$ is

$$\beta_n=\beta_0+r\cos(n\psi+\varphi),\quad \beta_0^2=r^2\cos^2\psi+\cot^2\psi. \tag{11}$$

Here r and φ are determined by the initial conditions. This violates the notion of "universality": asymptotics of the orbit under the renormalization depend on the parameters of the initial Hamiltonian.

6. Amplitude renormalization

The parameter map depends on the wavenumber k and the mass matrix through β. Consider first the case when $r=0$, so that $\beta=\cot(\psi)$ is fixed. In this case there are two fixed points, $A=B=C=0$ – the KAM fixed point, and the critical fixed point

$$A_c=\frac{2}{\sigma^{14}\beta},\quad B_c=\frac{2}{\sigma^8\beta},\quad C_c=\frac{2}{\sigma^{11}\beta}. \tag{12}$$

The KAM fixed point is stable. The stability of the critical point can be studied by taking the log of the amplitude map to give, in terms of $a=\log(A)$, $b=\log(B)$, $c=\log(C)$, the affine map

$$\begin{pmatrix}a'\\ b'\\ c'\end{pmatrix}=\begin{pmatrix}1&1&0\\ 0&0&1\\ 1&0&0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}+\begin{pmatrix}\log(\sigma^8\beta/2)\\ \log(\sigma^3)\\ \log(\sigma^3)\end{pmatrix}. \tag{13}$$

Thus stability is governed by the linear matrix above. This matrix has characteristic polynomial $\lambda^3-\lambda^2-1=0$ (interestingly, this polynomial is not related to the spiral mean), so that

$$\lambda_1=\delta\approx 1.465571232,$$

$$\lambda_{2,3}=\delta^{-1/2}e^{\pm i1.856478541}. \tag{14}$$

Thus there is a one dimensional unstable manifold, and a two dimensional, spiral stable manifold. The contraction on the stable manifold is rather slow, and the rotation number is very nearly $\frac{13}{44}$.

For the general case, β is not fixed, and the amplitude map is periodically forced. However, there is still a two dimensional center-stable manifold which has a one dimensional unstable manifold. On the center manifold the parameters converge to a circle on which the dynamics is a simple rotation with rotation number $\psi/2\pi$.

7. Conclusion

We have found an approximate renormalization description for the boundary of existence of the spiral mean torus for a $2\frac{1}{2}$ degree of freedom Hamiltonian – or equivalently a four dimensional symplectic map. The boundary is a codimension one surface in the space of parameters. It is the center-stable manifold of a critical fixed point of the renormalization operator with the single unstable eigenvalue $\delta\approx 1.4655$ and two neutral eigenvalues. All orbits on the center-stable manifold are attracted to the center manifold under renormalization. The renormalization dynamics on the center manifold is a rotation with irrational winding ratio.

Rotations arise because successive rational approximants of the incommensurate vector spiral inwards (there is an analogous oscillation in $1\frac{1}{2}$ degrees of freedom that is responsible for the momentum scaling eigenvalue being negative). This rotation gives rise to the rotation of the mass matrix parameters, which in turn drives an oscillation of the resonance amplitudes (A, B, C).

Thus, if we take our model at face value, it predicts that a typical one parameter system is not "self-similar" at criticality. Instead properties of the system such as the stability parameters of periodic orbits (i.e. the residues) are predicted to oscillate with rotation number of approximately $\frac{2}{9}$. The amplitude of the oscillation will depend upon the system studied; in our model it depends upon the off-diagonal element in the mass matrix (alternatively one can think of this as coming from the wavevectors not being perpendicular).

Indeed, previous attempts to find the critical point for a spiral mean torus have seen evidence for these oscillations. Artuso et al. [10] studied a 3D volume preserving map, and found that the residues of periodic approximations to a spiral mean torus oscillated, apparently with period 9.

Now the true renormalization dynamics need not look the same as our approximate model, even if it is a very good approximation. This is because no rotation is stable to perturbation. Arbitrarily small perturbations of a rotation can make the fixed point weakly attracting or repelling, and can generate weakly attracting or repelling invariant circles around the fixed point, or chains of periodic orbits, or Birkhoff attractors or worse! What *is* stable to perturbation, however, is the fixed point and a 2D normally hyperbolic invariant manifold containing it, with 1 unstable normal direction, the remaining normal directions being attracting. The stable manifold of this 2D normally hyperbolic manifold has codimension 1 and can be expected to be the boundary of KAM theory It would be worth trying to find the fixed point, because it would be an important handle on the normally hyperbolic manifold We call it a codimension-3 fixed point because in our model it has three eigenvalues which are not strictly inside the unit circle, so it has three-dimensional center-unstable manifold, and hence requires three parameters to find it.

The analogue of Greene's residue criterion might be used to find the fixed point by studying the stability of the periodic orbits making up successive cones in the Farey sequence for ω. Each of the three orbits bounding the cone has a pair of residues. One must find a set of parameter values for which the limit of all of these residues neither goes to infinity nor zero

We conjecture that the breakup boundary may have various components corresponding to the direct formation of full cantorus (which we know exists close enough to the anti-integrable limit for maps [23] or of a partial cantorus corresponding to a Cantor set cross a circle with various homotopy types. It is reasonable that the codimension three fixed point will form the organizing center for these various bifurcations.

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Appendix

Here we construct the renormalization transformation that corresponds to the Farey step "L". This transformation takes the cone formed from the resonances $M=I$ to $M=L^{-1}$. This corresponds to eliminating the $m_2=(0, 1, 0)$ resonance and adding the $m_2-m_1=(-1, 1, 0)$ resonance. To accomplish this we find coordinates to eliminate the $B\cos(2\pi ky)$ term in H. Supposing that the resonance amplitudes are formally $O(\epsilon)$, and that the frequency $m_2\cdot\omega=\omega_y\equiv k(\beta u+\gamma v)$ is nonzero on the orbits of interest (i.e. the neighborhood of the invariant torus), then this can be accomplished by the near identity canonical transformation $(x, y, u, v)\to(X, Y, U, V)$ generated by

$$S=Ux+Vy-\frac{B}{2\pi\omega_y}\sin(2\pi ky)+O(\epsilon^2), \tag{A.1}$$

so that $u=S_x=U+O(\epsilon^2)$, $v=S_y=V+O(\epsilon)$. This gives the Hamiltonian

$$H(X, Y, U, V)=\tfrac{1}{2}(U, V)\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\cdot\begin{pmatrix}U\\ V\end{pmatrix} +A\cos(2\pi x)+C\cos(2\pi lz)+\tfrac{1}{2}\gamma S_y^2+O(\epsilon^2) \tag{A.2}$$

Since $S_y^2=O(\epsilon^2)$, this term in H can be eliminated by adding an appropriate $O(\epsilon^2)$ to S. Substituting for $X=x+O(\epsilon)$, $Y=y+O(\epsilon)$, and $Z=t-X-Y$, and expanding the cosine terms gives four $O(\epsilon^2)$ resonance terms in H. We judiciously choose the $O(\epsilon^2)$ term in S (assuming that the resonance denominators, $\omega_x+\omega_y$ and $\omega_z\pm\omega_y$ are nonzero – as they will be in the neighborhood of our incommensurate torus) to eliminate all of these extra terms but one, leaving H in the form

$$H(X, Y, U, V)=\tfrac{1}{2}(U, V)\cdot\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\cdot\begin{pmatrix}U\\ V\end{pmatrix}$$
$$+A\cos(2\pi X)+C\cos(2\pi lZ)$$
$$+\frac{ABk\beta}{2\omega_y^2}\cos[2\pi(kY-X)]+O(\epsilon^3)\,. \tag{A.3}$$

The amplitude of the new resonance depends upon (U, V) through the resonance denominator ω_y. In the spirit of Chirikov we evaluate this amplitude on the $(-1, 1, 0)$ resonance, thus

$$0=k\dot{y}-\dot{x}\quad\rightarrow\quad\omega_y=\omega_x\,.$$

However, this does not determine the frequency. We choose, somewhat arbitrarily, to evaluate the frequency at the new periodic orbit, $p=[1, 1, 0]$, where $0=\omega_z=\dot{z}=1-\dot{x}-\dot{y}$. This implies that

$$\omega_y=\omega_x=\frac{k}{1+k}\,. \tag{A.4}$$

We now transform the coordinates on the torus as in (4), to return the phases in H to their original form. This transformation $(X, Y, U, V)\rightarrow(x', y', u', v')$ is generated by

$$S=u'(kY-X)+kv'(t-X-Y)\,, \tag{A.5}$$

so that

$$\begin{pmatrix}U\\ V\end{pmatrix}=\begin{pmatrix}-1 & -k\\ k & -k\end{pmatrix}\begin{pmatrix}u'\\ v'\end{pmatrix}=W\begin{pmatrix}u'\\ v'\end{pmatrix}. \tag{A.6}$$

Note that even if the mass matrix were originally diagonal, this transformation would generate an off-diagonal term. The new Hamiltonian is

$$H=\tfrac{1}{2}(u', v')\cdot W^{t}\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}W\cdot\begin{pmatrix}u'\\ v'\end{pmatrix}+kv'$$
$$+\frac{AB\beta k}{2\omega_y^2}\cos(2\pi x')+C\cos(2\pi k'y')$$
$$+A\cos(2\pi l'z')+O(\epsilon^3)\,, \tag{A.7}$$

where $z'=kt-x'-y'$. The linear term kv' can be absorbed by shifting the origin of the momenta. Finally, we rescale time to restore the form of z, rescale the momenta to make the determinant of the new mass matrix unity, and rescale H to maintain the canonical form of the equations:

$$t=kt,\quad(\bar{u}, \bar{v})=(1+k)(u', v'),\quad\bar{H}=\frac{1+k}{k}H\,. \tag{A.8}$$

The resulting Hamiltonian is given by (1) to $O(\epsilon^3)$, under the map (5), (6). The final momentum transformation, including the shift in origin, is

$$u'=-u+v-\beta'+O(\epsilon)\,,$$
$$v'=-u-\frac{1}{k}v+\alpha'+O(\epsilon)\,. \tag{A.9}$$

Note that this is an expanding map on the momenta (when $k>0$), corresponding to enlarging the phase space in the neighborhood of the torus.

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