Review of the big picture on polynomial approximation.

- We are studying how to approximate functions $f \in C[a, b]$ using polynomials and trig polynomials.
- First you have to pick a norm and a polynomial degree n. Then you're trying to find a polynomial of degree $\leq n$ that minimizes the norm of the error. Such a polynomial (or trig polynomial) always exists, no matter what norm you're using.
- We started with L^∞, and talked about existence & uniqueness, construction & convergence. Weierstrass & Cheby EquiOscillation are main results.
- We then moved to weighted L^2 norms. Once you pick a norm, the optimal polynomial exists and is unique. Also, it's guaranteed to converge in L^2 as the degree of the polynomial increases.
- There are a variety of ways to compute/construct the L^2-optimal polynomial; in general you pick a basis for the space of polynomials of limited degree, then solve a linear system for the coefficients.
- The standard monomial basis is often inconvenient because the linear system can be large and ill-conditioned. It's convenient to use a basis for the space of polynomials that is orthogonal with respect to the weighted L^2 inner product associated with the norm you're using, because then you can compute the solution without solving a linear system of equations.
- We talked about various families of orthogonal polynomials, and the properties that they all share (three-term recurrences, roots, Clenshaw summation).
- We looked at trig polynomial approximations in L^2. If the function $f \in C^\infty$ and periodic, then the Fourier series and all its derivatives will converge not just in L^2 but absolutely and uniformly. (I didn't show this completely, jst heuristically; you'll see it in Applied Analysis and/or PDEs.)
- We showed that Chebyshev polynomials are just Fourier series after a change of variable. If $f \in C^\infty[-1, 1]$ then the Chebyshev series (and all its derivatives) will converge absolutely and uniformly because it's just a Fourier series and the nonlinear change of variable is infinitely smooth.
- The above argument shows that it's a good idea to use a particular weighted norm: the one associated with Chebyshev polynomials. If you use a different norm then you don't get the same nice convergence guarantees. It's possible to prove things about other weight functions, but in general the results are (i) not as good and (ii) harder to prove.