Remember to write your name! You are allowed to use a calculator. **You are not allowed to use the textbook or your notes or your neighbor**. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise. You may cite any theorem from Atkinson or from the lectures unless explicitly stated otherwise.

You must do the first problem. You must pick **only** two of the remaining problems. Each problem is 15 points; there are 45 points total.

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1. Quadrature Consider a quadrature of the form

$$\int_{-h}^{h} f(x)\mathrm{d}x \approx h(af(-h) + bf(0) + cf(h)) + dh^2 f'(0).$$

(a) Find coefficients so that the quadrature integrates all cubic polynomials exactly.

(b) The above quadrature is equivalent to using Hermite interpolation with the available data, and the quadrature error is therefore

$$\int_{-h}^{h} f(x)\mathrm{d}x - h(af(-h) + bf(0) + cf(h)) - dh^2 f'(0) = \int_{-h}^{h} \frac{(x^2 - h^2)x^2}{4!} f^{(4)}(\xi(x))\mathrm{d}x.$$

Use this to find an asymptotic formula for the error in the limit $h \to 0$.

(c) Consider a composite quadrature for integrals on $[a, b]$ based on repeated use of the above formula. Use the asymptotic error estimate derived above to form a 'corrected' composite quadrature similar to the corrected Trapezoid Rule, assuming you know the values of $f^{(3)}(a)$ and $f^{(3)}(b)$.

2. Linear Systems Suppose that the LU factorization of $\mathbf{A}$ can be computed without pivoting. Let

$$\mathbf{A} = \begin{bmatrix} \hat{\mathbf{A}} & \boldsymbol{d} \\ \boldsymbol{c}^T & \alpha \end{bmatrix}, \quad \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^{n-1}, \quad \alpha \in \mathbb{R}$$

and assume that $\hat{\mathbf{A}} = \hat{\mathbf{L}}\hat{\mathbf{U}}$, the LU-factorization of $\hat{\mathbf{A}}$, has already been computed. Explain how to use this to compute the LU factorization of $\mathbf{A}$. (Note that this can be used to recursively compute an LU factorization by computing LU factorizations of the upper-left $k \times k$ blocks with increasing k.)

3. Rootfinding/Nonlinear Equations Consider the fixed-point iteration $x_{k+1} = 2^{x_k - 1}$. (Recall $2^x = e^{x \ln(2)}$.)

(a) What are the fixed points?

(b) Which fixed points are stable, i.e. locally convergent?

(c) For each fixed point, what is the order of convergence (i.e. linear, quadratic, etc)?

(d) Formulate Newton's method for $f(x) = x - 2^{x-1}$. What is the order of convergence for each root?

4. Interpolation

(a) Let $x_1, \ldots, x_n$ be n distinct points, and suppose that $p_{n-1}(x)$ interpolates $f(x)$ at $x_1, \ldots, x_{n-1}$ while $q_{n-1}(x)$ interpolates $f(x)$ at $x_2, \ldots, x_n$ (p and q are polynomials of degree $\leq n-1$). Show that

$$p_n(x) = \frac{(x - x_1)q_{n-1}(x) - (x - x_n)p_{n-1}(x)}{x_n - x_1}$$

is a polynomial of degree $\leq n$ that interpolates $f(x)$ at $x_1, \ldots, x_n$.

(b) The above fact can be used to develop a recursive algorithm for finding an interpolating polynomial. Consider the following table

$$\begin{array}{cc|cc} x_1 & f_1 & & \\ x_2 & f_2 & P_{1,2}(x) & \\ x_3 & f_3 & P_{2,3}(x) & P_{1,2,3}(x) \end{array}$$

where $P_{1,2}(x)$ interpolates $f(x)$ at points x_1 and x_2, $P_{2,3}(x)$ interpolates at x_2 and x_3, and $P_{1,2,3}(x)$ interpolates at all the points. Assume that the nodes are $-1, 0, 1$ and values are $f(-1) = -1$, $f(0) = 0$, $f(1) = 1$, and give explicit expressions for all 3 P polynomials in the table above using the recursion from (a).

(c) The above table can also be used to compute *values* of the interpolating polynomial. Add another point $x_4 = 2$ with value $f_4 = f(2) = 8$. Without computing the new interpolating polynomial, use the table to compute the *value* of the interpolating polynomial at $x = 1/2$.

5. Approximation

(a) Find the coefficients a_j and b_j to minimize

$$\int_{-\pi}^{\pi} \left(\cos((n+1)x) - \left[a_0 + \sum_{j=1}^{n} a_j \cos(jx) + b_j \sin(jx) \right] \right)^2 \mathrm{d}x.$$

(b) Let $p(x)$ be the trig polynomial of degree at most n that minimizes $\int_{-\pi}^{\pi}(f(x) - p(x))^2 \mathrm{d}x$ for some periodic, continuously-differentiable f. Show that $q(x) = p'(x)$ is the trig polynomial of degree at most n that minimizes $\int_{-\pi}^{\pi}(f'(x) - q(x))^2 \mathrm{d}x$.

(c) Let $p(x)$ be the polynomial of degree at most $n \geq 1$ that minimizes $\int_{-1}^{1}(f(x) - p(x))^2 \mathrm{d}x$ for some continuously-differentiable f. Show by example that the polynomial $q(x)$ of degree at most $n - 1$ that minimizes $\int_{-1}^{1}(f'(x) - q(x))^2 \mathrm{d}x$ is not necessarily $p'(x)$.