Density theorems

<u>Thm 12.48</u> Simple functions are dense in L^p, $1 \le p \le \infty$
i.e. $\forall f \in L^p \ \exists \ (\varphi_n)$ simple s.t. $\|f - \varphi_n\|_p \to 0$.

<u>proof</u> ($p \neq \infty$ case only)
Split up $f = f_{\mathbb{R}} + i f_{img}$, $f_{\mathbb{R}} = f_+ - f_-$, prove for each component and use Δ-inequality
Hence assume $f \in \mathbb{R}$ + $f \ge 0$, wlog.

$f \ge 0 \Rightarrow \exists$ monotone increasing simple fcn φ_n st. $\varphi_n \xrightarrow{pw} f$, $\varphi_n \nearrow f$.
Similarly, $\varphi_n{}^p \nearrow f^p$.
Then

$$\lim_{n\to\infty} \|f - \varphi_n\|_p{}^p := \lim \int |f - \varphi_n|^p = \lim \int \underbrace{(f - \varphi_n)^p}_{\text{decreasing}} \quad \text{since } f \ge \varphi_n$$

$$= \int f^p - \lim_{n\to\infty} \int \underbrace{f^p - (f - \varphi_n)^p}_{\text{increasing}}$$

$$\overset{\text{M.C.T.}}{=} \int f^p - \int \left[f^p - \lim (f - \varphi_n)^p \right] = 0. \quad \square$$

<u>Corollary</u>: Thm 12.49, L^p is separable for $1 \le p < \infty$ (<u>not</u> true for $p = \infty$).
proof sketch:
1) use simple fcn (from prev. thm) to form a dense set (but not yet countable)
2) simple fcn w/ rational coefficients
3) approximate measurable sets A_i w/ all intervals w/ rational end-pts.
(this fails to work for $p = \infty$ case)

<u>Fact</u>: L^∞ isn't separable

<u>Thm 12.50</u> For $1 \le p < \infty$, $\overline{C_c(\mathbb{R}^n)} = L^p(\mathbb{R}^n)$
proof. Fix $f \in L^p$
1) define $f_n(x) = f(x) \cdot \chi_{[-n,n]}$, so compactly supported
and $\exists$ n st. $\|f_n - f\|_p < \varepsilon$ (via D.C.T., since $f_n^p \le f^p \in L^1(\mathbb{R}^n)$)
(This step fails w/ $p = \infty$)
2) approximate f_n via simple fcn, as above
3) each simple fcn is finite sum of χ_A (and A is a bdd set, by 1) above)
4) <u>for A bdd. + msbl., can approximate χ_A in C_c</u>

proof on next page

(proof cont'd)

A is msbl (wrt Lebesgue) means $\exists$ open G and closed K s.t.

$K \subseteq A \subseteq G$ and $\lambda(G \setminus K) < \varepsilon$

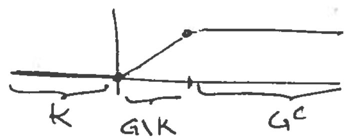
Via Urysohn's lemma (exerc 1.16), for any $\overset{\text{closed}}{K} \subseteq \overset{\text{open}}{G} \subseteq \mathbb{R}^n$,

$\exists$ continuous fcn $f: \mathbb{R}^n \to \mathbb{R}$ s.t. $0 \le f(x) \le 1$ and

1) $f(x) = 1$ on K

2) $f(x) = 0$ on G^c, e.g. $f(x) = \dfrac{d(x, G^c)}{d(x, G^c) + d(x, F)}$

example, $K = (-\infty, 0]$
$G = (-\infty, 1)$ so $G^c = [1, \infty)$



$d(x, G^c) = 1 - x$ for $x \in [0,1]$
$d(x, K) = x$ for $x \in [0, 1)$
so $f(x) = \dfrac{1-x}{(1-x)+x} = 1 - x$.

So using this $f \in C$
(in fact $f \in C_c$), $\|f - \chi_A\|_p^p = \int_{G \setminus K} |f - \chi_A|^p \le 1 \cdot \lambda(G \setminus F) < \varepsilon$
hence dense.

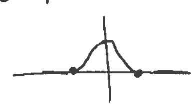
5) Finite sum of compactly supported continuous functions is still in C_c. $\square$

Improvement:
Thm 12.51 In fact, $1 \le p < \infty$, $\overline{C_c^\infty(\mathbb{R}^n)} = L^p$

proof sketch
continue as above, after showing C_c is dense in L^p

6) $\forall \varepsilon > 0$, let $\varphi_\varepsilon \in C_c^\infty$ be an approx. identity, eg. positive mollifier,

such as $\varphi(x) = \begin{cases} e^{-1/1+|x|^2} & |x| < 1 \\ 0 & |x| \ge 1 \end{cases}$



Define $f_\varepsilon = \varphi_\varepsilon * f \in C_c^\infty$ and $f_\varepsilon \to f$ uniformly

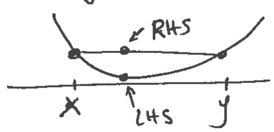
7) Since we're working w/ f_n w/ compact support, uniform convergence $\Rightarrow$ L^p convergence. $\square$

Basic Inequalities, §12-7

Use convexity a lot:

__Define__ $\varphi : C \to \mathbb{R}$, C a convex set, to be a __convex function__ if

$$\forall\, t \in [0,1] \text{ and } \forall\, x,y \in C, \quad \varphi(\underbrace{t\cdot x + (1-t)\cdot y}_{\text{LHS}}) \le \underbrace{t\cdot\varphi(x) + (1-t)\varphi(y)}_{\text{RHS}}$$



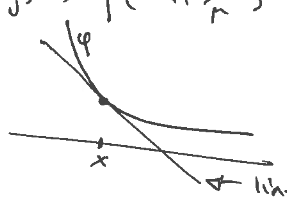
Notation:
the __mean__ of $f \in L^1$ on a finite measure space is $\langle f \rangle_\mu := \frac{1}{\mu(X)} \int_X f\, d\mu$.

__Thm 12.52 Jensen__ let (X,μ) be a finite msr space, eg. a probability space.
If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and $f \in L^1(X)$, then

$$\varphi(\langle f \rangle_\mu) \le \langle \varphi \circ f \rangle_\mu \qquad (\text{if } \mu(X)=1,\ \langle f \rangle_\mu = \mathbb{E}(f))$$

(example: $\varphi(t)=t^2$ is convex, so if $\mu(X)=1$, $(\mathbb{E}f)^2 \le \mathbb{E}(f^2)$)

__proof__
1) via exercise 12.9, at any point (eg. $x = \langle f \rangle_\mu$), $\exists\, c \in \mathbb{R}$
st. $\forall y$, $\varphi(y) \ge \varphi(\langle f \rangle_\mu) + c\cdot(y - \langle f \rangle_\mu)$



Note: we say "c" is a subgradient.

2) set $y = f(x)$,

$$\int \varphi\circ f\, d\mu \ge \int \left(\varphi(\langle f\rangle) + c\cdot(f - \langle f \rangle) \right) d\mu$$

$$\ge \varphi(\langle f \rangle)\cdot \int d\mu + \underbrace{c\cdot\int f\, d\mu - c\cdot\langle f\rangle \int d\mu}_{\text{cancel}}$$

$$= \mu(X)\cdot\varphi(\langle x \rangle) \quad \square$$

__Thm: Hölder's Inequality__
Let $f \in L^p(X,\mu)$ for $1 \le p \le \infty$ and let p' be the Hölder conjugate of p, ie. $\frac{1}{p} + \frac{1}{p'} = 1$
and $g \in L^{p'}(X,\mu)$, then
1) $f\cdot g \in L^1(X,\mu)$
2) $\left| \int fg\, d\mu \right| \le \|f\|_p \cdot \|g\|_{p'}$

and we can also deduce

3) $\int |fg|\, d\mu \le \|f\|_p \cdot \|g\|_{p'}$ (replace f with $|f|$, and $f \in L^p \Leftrightarrow |f| \in L^p$).

proof: see book

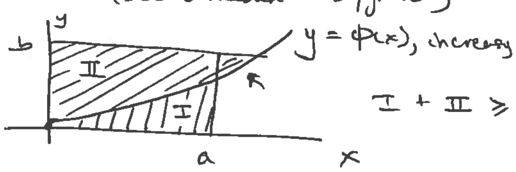
Thm. Young's Inequality

let $y=\phi(x)$ be cts and strictly increasing for $x \geq 0$ with $\phi(0)=0$
($\phi: \mathbb{R} \to \mathbb{R}$)

let $\psi = \phi^{-1}$ so $x = \psi(y)$. Then $\forall\, a, b > 0$,

$$a \cdot b \leq \underbrace{\int_0^a \phi(x)\,dx}_{\mathrm{I}} + \underbrace{\int_0^b \psi(y)\,dy}_{\mathrm{II}} .$$ Equality iff $b = \phi(a)$.

proof by picture: (see Wheeden + Zygmund)



$\mathrm{I} + \mathrm{II} \geq$ rectangle's area $= a \cdot b$.

Corollary (sometimes also known as Young's)

let $\phi(x) = x^\alpha$, $\alpha > 0$, so $\psi(y) = y^{1/\alpha}$, then

1) $ab \leq \dfrac{a^{1+\alpha}}{1+\alpha} + \dfrac{b^{1+1/\alpha}}{1+1/\alpha}$, and letting $p = \alpha+1$, $p' = \frac{1}{\alpha}+1$,

2) $ab \leq \dfrac{a^p}{p} + \dfrac{b^{p'}}{p'}$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$

3) "Peter-Paul" inequality (exer. 12.10)

$$ab \leq \varepsilon \cdot \frac{a^2}{2} + \frac{1}{\varepsilon} \cdot \frac{b^2}{2}.$$

Proof of Hölder via Young's

Case 1: $p=1$, $p' = \infty$ (or vice-versa)

$$\int |fg|\,d\mu \leq \|f\|_\infty \int_E |g|\,d\mu + \cancel{\int_N |fg|\,d\mu}$$

where $f(x) \leq \|f\|_\infty$ on E (hence $\mu(N) = 0$)

$$= \|f\|_\infty \|g\|_1 .$$

Case 2: $p, p' > 1$.

Trick: wlog, normalize so that $\|f\|_p = \|g\|_{p'} = 1$ (we'll scale back later)

$$\int |fg| = \int \underbrace{|f|}_{a} \cdot \underbrace{|g|}_{b} \underset{\uparrow}{\leq} \int \frac{|f|^p}{p} + \frac{|g|^{p'}}{p'} = \frac{1}{p}\|f\|_p^{\,p} + \frac{1}{p'}\|g\|_{p'}^{\,p'}$$

Young's ineq., ptwise $\quad = \frac{1}{p} \cdot 1 + \frac{1}{p'} \cdot 1 = 1 = \|f\|_p \cdot \|g\|_{p'}$ ◻

Corollary ($p = p' = 2$)

Cauchy-Schwarz.

Thm "converse to Hölder" (Thm 8.8 Wheeden + Zygmund)

If $f:(X,\mu)\to\mathbb{R}$ is a measurable function, and $\frac{1}{p}+\frac{1}{p'}=1$ for $1\le p\le\infty$,

then $\|f\|_p = \sup_{\|g\|_{p'}\le 1} \int f\cdot g$, i.e. $\|f\|_p$ is an operator norm, even $p=\infty$.

proof: $\|f\|_p \ge \sup_{\|g\|_{p'}\le 1} \int f\cdot g$ via Hölder's.

For brevity (these are technical), exclude $\|f\|_p = 0$ or ∞,

take $g = f^{p/p'}$ so $\int f\cdot g = \|f\|_p$ and $\|g\|_{p'}=1$. □

Implications of Hölder

Prop. 12.55 If $\mu(X)<\infty$, then $L^p(X,\mu)\subseteq L^q(X,\mu)$ if $p\ge q$

i.e. $L^\infty \subseteq L^p \subseteq L^q \subseteq L^1$ (eg. $X=\mathbb{T}$ w/ $\mu=\lambda$).

proof see book, straightforward.

Not true if $\mu(X)=\infty$, eg. $\mu=\lambda$ on $\mathbb{R}$, $f(x)=\frac{1}{x}$, $f\in L^2\setminus L^1$
$X=[1,\infty)$

we do have an interpolation result:

If $p<r<q$, then $L^p\cap L^q\subseteq L^r$

(ie. If $f\in L^1$, its not true $f\in L^2$. But $f\in L^1$ and $f\in L^\infty \Rightarrow f\in L^2$).

Thm 12.56 Minkowski (Δ-ineq on L^p spaces)

$1\le p\le\infty$, $f,g\in L^p(X,\mu)$, then
1) $f+g\in L^p$
2) $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

proof: see book.

Thm 12.58 Young, version 2

$1\le p,q,r\le\infty$ w/ $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$ ($r=\infty$ ok), $X=\mathbb{R}^n$

then $\begin{matrix} f\in L^p \\ g\in L^q\end{matrix} \Rightarrow (f*g)\in L^r$ and $\|f*g\|_r \le \|f\|_p\,\|g\|_q$.

proof: see book.

§12.8 Dual Spaces of L^p

(the space and measure (X,μ) will be implicit)

We'll use the "duality pairing" $\langle f,g\rangle := \int fg\,d\mu$

Thm 12.59 $1<p<\infty$ (need to exclude $p=1,\infty$), then $(L^p)^* \cong L^{p'}$ for $\frac{1}{p}+\frac{1}{p'}=1$.

Proof (not in book)

Clearly $L^{p'} \subseteq (L^p)^*$ under the embedding

$$g \mapsto \varphi_g, \quad \varphi_g : L^p \to \mathbb{R} \text{ by } \varphi_g(f) = \langle f,g\rangle$$

since φ_g is 1) linear
2) bdd (via Hölder's ineq.).

To show the converse, follow Reed & Simon (Thm 5.4 p 350):

1) φ is a finite linear combo of positive operators ψ s.t. if $f \geq 0$, $\psi(f) \geq 0$.

2) let $\mu(X)<\infty$, though this can be relaxed to σ-finite.

3) define a measure $\nu(A) = \varphi(\chi_A)$, which is a valid measure if $p<\infty$ and φ is a pos. operator. Use MCT.

4) via Radon-Nikodym thm, $\exists\, g \in L^1$ st $\varphi(\chi_A) = \int g \cdot \chi_A\,d\mu$

5) use MCT, a few more steps. □

If X is σ-finite, require absolute continuity, but not if $\mu(X)<\infty$
see p. 344.

Hence we can talk about weak convergence.

- If $1 \leq p < \infty$, $(f_n) \subseteq L^p$ converges weakly $f_n \rightharpoonup f$ if $\forall g \in L^{p'}$ $(\frac{1}{p}+\frac{1}{p'}=1)$, $\lim\limits_{n\to\infty} \int f_n g\,d\mu = \int fg\,d\mu$.
- If $p=\infty$, if $\forall g \in L^{p'} = L^1$, $\lim \int f_n g = \int fg$, we only have weak-* convergence. $f_n \overset{*}{\rightharpoonup} f$.

Ex Fix any $g \in L^p$, $1<p<\infty$, $g \neq 0$. The following $(f_n) \subseteq L^p$ converge weakly but not strongly to 0.

(A) Oscillation. $f_n(x) = g(x)\cdot \sin(nx)$
(B) Concentration. $f_n(x) = n^{1/p}\cdot g(nx)$
(C) Escape to Infinity. $f_n(x) = g(x-n)$.

Recall $f_n \rightharpoonup f$ iff $\begin{cases} T(f_n) \to T(f) \ \forall\, T \in \text{dense subset of } (L^p)^* \\ \text{AND} \\ \|f_n\| \leq c \end{cases}$

Thm Corollary of Banach-Alaoglu.
$(f_n) \subseteq L^p$ is bounded, $1<p<\infty$, then there is a weakly convergent subsequence.