Density Theorems

Theorem 12.48  Simple functions are dense in $L^p$, $1 \leq p < \infty$.

Proof: (p < \infty case only)
Split up $f = f^n + f^-$, $f^n = f^n_n - f^n_1$, prove for each component
and use triangle inequality.
Hence assume $f \geq 0$, wlog.

$f > 0 \Rightarrow \exists$ monotone increasing simple func $\phi_n$ st. $\phi_n \to f$, $\phi_n \to f^+$.
Similarly, $\phi_n \to f^-$.

Then
\[
\lim_{n \to \infty} \|f - \phi_n\|_p = \lim_{n \to \infty} \int (f - \phi_n)^p = \int (f - \phi_n)^p \quad \text{decreasing}
\]
\[
= \int f^p - \lim_{n \to \infty} \int (f - \phi_n)^p \quad \text{increasing}
\]

M.C.T.: $\int f^p - \int [f^p - \lim (f - \phi_n)^p] = 0$. \( \square \)

Corollary: Theorem 12.49, $L^p$ is separable for $1 \leq p < \infty$ (not true for $p = \infty$).

Proof sketch:
1) Use simple fun (from prev. thm) to form a dense set (but not yet countable)
2) Simple fun w/ rational coefficients
3) Approximate measurable sets $A$, w/ all intervals $I$, rational end pts.
   (this fails to work for $p = \infty$ case)

Fact: $L^\infty$ isn't separable.

Theorem 12.50  For $1 \leq p < \infty$, $C_c(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

Proof: Fix $f \in L^p$

1) Define $f_n(x) = f(x) \chi_{[n,n+1]}$, so compactly supported
   and $\int f_n = \int f$ (by D.C.T., since $f_n \to f \in L^p(\mathbb{R}^n)$)
   (This step fails w/ $p = \infty$)

2) Approximate $f_n$ via simple fun, as above
3) Each simple fun is finite sum of $\chi_A$ (and $A$ is a ball set, by 1 above)
4) For a ball w/ midpt, can approximate $\chi_A$ in $C_c$

Proof on next page
(proof cont'd)

A is usbd (wrt lebesgue) means \( \exists \) open \( G \) and closed \( K \) st.

\[ K \subseteq G \subseteq \mathbb{R}^n \text{ and } \lambda(G \setminus K) < \epsilon \]

Via Dieudonné's lemma (exrc 1.16), for any \( K \subseteq G \subseteq \mathbb{R}^n \),

\( \exists \) continuous \( f : \mathbb{R}^n \to \mathbb{R} \) st. \( 0 \leq f(x) \leq 1 \) and

1) \( f(x) = 1 \) on \( K \)
2) \( f(x) = 0 \) on \( G^c \), e.g. \( f(x) = \frac{d(x, G^c)}{d(x, G) + d(x, F)} \)

\begin{itemize}
  \item example, \( K = (-\infty, a] \)
  \item \( G = (-\infty, 1) \) so \( G^c = [1, \infty) \)
\end{itemize}

\[ d(x, G^c) = 1-x \text{ for } x \in [0, 1] \]
\[ d(x, K) = x \text{ for } x \in [0, 1] \]
\[ \text{so } f(x) = \frac{1-x}{(1-x)+x} = 1-x. \]

So using this \( f \in C^0 \)

(\text{in fact } f \in C^\infty_c), \( \| f - \chi_k \|_p \leq \int_{G \setminus K} |f - \chi_k|^p = 1 \cdot \lambda(G \setminus K) < \epsilon \)

hence dense.

5) Finite sum of compactly supported continuous functions is still in \( C^0 \). □

Improvement:

Then 12.5! In fact, \( 1 \leq p < \infty, \ C^\infty_c(\mathbb{R}^n) = L^p \)

\text{proof sketch}

\text{continues as above, after showing } C^\infty_c \text{ is dense in } L^p

6) \( \forall \epsilon > 0, \ \exists \phi_c \in C^\infty_c \text{ be an approx. identity, e.g. positive mollifier, such as } \phi_c(x) = \begin{cases} e^{-1/10|x|^2} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \)

\text{Define } f_c = \phi_c \ast f \in C^\infty_c \text{ and } f_c \to f \ \text{uniformly}

7) Since we're working w/ \( f \) \( \| f_n \|_p \text{ compact support, } \)

\text{uniform converg } \implies \text{ } L^p \text{ converg. } □
Basic Inequalities, § 12.7

Use convexity a lot:

Define \( \varphi : C \to \mathbb{R} \), \( C \) a convex set, to be a convex function if

\[
\forall t \in [0,1] \text{ and } \forall x, y \in C, \quad \varphi(t \cdot x + (1-t) \cdot y) \leq t \cdot \varphi(x) + (1-t) \cdot \varphi(y)
\]

Notation:

the mean of \( f \in L^1 \) on a finite measure space is \( \langle f \rangle_\mu := \frac{1}{\mu(X)} \int_X f \, d\mu \).

Thm 12.52 Jensen

Let \( (X, \mu) \) be a finite measure space, e.g., a probability space.

If \( \varphi : \mathbb{R} \to \mathbb{R} \) is convex and \( f \in L^1(X) \), then

\[
\varphi\left( \langle f \rangle_\mu \right) \leq \langle \varphi \circ f \rangle_\mu \quad \left( \text{if } \mu(X) = 1, \quad \langle f \rangle_\mu = \mathbb{E}(f) \right)
\]

Example: \( \varphi(t) = t^2 \) is convex, so if \( \mu(X) = 1 \),

\[
(\mathbb{E}(f^2))^2 \leq \mathbb{E}(f^2)^2
\]

Proof:

via exercise 12.9, at any point (e.g., \( x = \langle f \rangle_\mu \)), \( f \in \mathbb{R} \)

st. \( \forall y, \; \varphi(y) \geq \varphi(\langle f \rangle_\mu) + \varphi'(y - \langle f \rangle_\mu) \cdot (y - \langle f \rangle_\mu) \)

Note: we say "c" is a subgradient.

2) Set \( y = \varphi(x) \),

\[
\int \varphi \circ f \, d\mu \geq \int \left( \varphi(\langle f \rangle_\mu) + c \cdot (f - \langle f \rangle_\mu) \right) \, d\mu
\]

\[
= \varphi(\langle f \rangle_\mu) \cdot \int f \, d\mu + c \cdot \int f \, d\mu - c \cdot \langle f \rangle_\mu \cdot \mu(X) \quad \square
\]

Thm: Hölder’s Inequality

Let \( f \in L^p(X, \mu) \) for \( 1 \leq p < \infty \) and let \( p' \) be the Hölder conjugate of \( p \), i.e., \( \frac{1}{p} + \frac{1}{p'} = 1 \)

1) \( \varphi \in L^{p'}(X, \mu) \)

2) \( | \int fg \, d\mu | \leq \| f \|_p \cdot \| g \|_{p'} \)

and we can also deduce

3) \( \int |fg| \, d\mu \leq \| f \|_p \cdot \| g \|_{p'} \) (replace \( f \) with \( |f| \) and \( g \) with \( |g| \)).

Proof: see book.
Thm. Young's Inequality

Let \( y = \phi(x) \) be cts and strictly increasing for \( x \geq 0 \) with \( \phi(0) = 0 \)
(\( \phi: \mathbb{R} \to \mathbb{R} \))

Let \( \psi = \phi^{-1} \) so \( x = \psi(y) \). Then for all \( a, b > 0 \),

\[
 a \cdot b \leq \int_a^b \phi(x) \, dx + \int_b^a \psi(y) \, dy. \quad \text{Equality iff } b = \phi(a).
\]

Proof by picture: (see Wheeden & Zygmund)

\[ a \quad b \quad x \quad y \]

\[ \text{I + II } \geq \text{rectangle's area } = a \cdot b. \]

Corollary (sometimes also known as Young's)

Let \( \psi(x) = x^p, \ x > 0 \), so \( \psi'(y) = y^{p-1} \), then

1) \( ab \leq \frac{a^{1+\frac{1}{p}}}{1+\frac{1}{p}} + \frac{b^{1+\frac{1}{q}}}{1+\frac{1}{q}} \), and letting \( p = x+1, \ p' = \frac{1}{x+1}, \)

2) \( ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \ 1 < p < x, \ \frac{1}{p} + \frac{1}{q} = 1 \)

3) "Peter-Paul" Inequality (exer. 12.16)

\( ab \leq \varepsilon \cdot \frac{a^2}{2} + \frac{1}{\varepsilon} \cdot \frac{b^2}{2} \).

Proof of Hölder via Young's

Case 1: \( p = 1, \ p' = \infty \) (or vice versa)

\[
\int f \cdot g \, dp \leq \|f\|_{L^p} \|g\|_{L^{p'}} \int f \cdot g \, dp
\]

where \( f(x) = \|f\|_{L^p} \) on \( E \) (hence \( \mu(E) = 0 \))

\( = \|f\|_{L^p} \|g\|_{L^{p'}} \).

Case 2: \( p, p' > 1 \).

Trick: Why, normalize so that \( \|f\|_p = \|g\|_{p'} = 1 \) (will scale back later)

\[
\|fg\|_{L^\infty} \leq \int f \cdot g \leq \int |f|^p + |g|^{p'} = \frac{1}{p} \|f\|_p^p + \frac{1}{p'} \|g\|_p^{p'}
\]

Yong's Ineq., proof \( = \frac{1}{p} + \frac{1}{p'} = 1 = \|f\|_p \cdot \|g\|_{p'}/\Delta \)

Corollary (\( p=p'=2 \))

Caucluschwarz.
Converse to Hölder (Thm 8.8 Wheeden & Zygmund)

If \( f : (X, \mu) \to \mathbb{R} \) is a measurable function, and \( \frac{1}{p} + \frac{1}{q} = 1 \) for \( 1 \leq p \leq \infty \),
then
\[
\|f\|_p = \sup_{\|g\|_q \leq 1} \int f \cdot g, \quad \text{ir.} \quad \|f\|_p \text{ is an operator norm,}
\]
for \( p = \infty \).

**Proof**: \( \|f\|_p \geq \sup_{\|g\|_q \leq 1} \int f \cdot g \) via Hölder's.

For brevity (these are technical), exclude \( \|f\|_p = 0 \) or \( \infty \),
take \( g = f^{\prime \prime} \), so \( \int f \cdot g = \|f\|_p \) and \( \|g\|_p = 1 \).

Implications of Hölder

**Prop 12.55**
If \( \mu(X) < \infty \), then \( L^p(X, \mu) \subseteq L^q(X, \mu) \) if \( p \geq q \)
\( \text{i.e.} \ L^\infty \subseteq L^p \subseteq L^q \subseteq L^1 \) (eg. \( X = \mathbb{T} \), \( \mu = \lambda \)).

**Proof**: see book, straightforward.

Note: true if \( \mu(X) = \infty \), e.g. \( \mu = \lambda \) on \( \mathbb{R} \), \( f(x) = \frac{1}{x} \), \( f \in L^2 \setminus L^1 \)
\( X = (1, \infty) \).

We do have an interpolation result:
if \( p < r < q \), then \( L^p \cap L^q \subseteq L^r \)
(ie. if \( f \in L^1 \), it's not true \( f \in L^2 \). But \( f \in L^1 \) and \( f \in L^\infty \implies f \in L^2 \).

**Thm 12.56** Minkowski (A. ineq on \( L^p \) space)

\( 1 \leq p \leq \infty \), \( f, g \in L^p(X, \mu) \), then
1) \( f + g \in L^p \)
2) \( \|f + g\|_p \leq \|f\|_p + \|g\|_p \).

**Proof**: see book.

**Thm 12.58** Young, version 2

\( 1 \leq p, r \leq \infty \) w/ \( \frac{1}{r} + \frac{1}{r'} = 1 + \frac{1}{r'} \) (c = \infty ok), \( X = \mathbb{R}^n \)
then \( f \in L^p \) \Rightarrow \( f^r \in L^r \) and \( \|f^r\|_r \leq \|f\|_p \|g\|_r \).

**Proof**: see book.
§ 12.8 Dual Spaces of $L^p$

The space and measure $(X, \mu)$ will be implicit.

We'll use the "duality pairing" $\langle f, g \rangle = \int f g \, d\mu$.

Theorem 12.59. \(1 < p < \infty\) (need to exclude \(p = 1, \infty\)),

then \(L^p(X)^* \equiv L^q(X)\) for \(\frac{1}{p} + \frac{1}{q} = 1\).

Proof (not in book).

Clearly \(L^p(X)^* \subseteq (L^p(X))^*\) under the embedding \(g \mapsto \phi_g, \phi_g : L^p(X) \to \mathbb{R}\) by \(\phi_g(f) = \langle f, g \rangle\).

Since \(\phi_g\) is \(\mathcal{B}\)-linear,

2) holds (via Hölder's Ineq.).

To show the converse, follow Reed & Simon (Thm 5.4, p 350):

1) \(\phi\) is a finite linear combo of positive operators.

2) \(\|\phi(x)\| < \infty\), though this can be relaxed to \(\sigma\)-finite.

3) define a measure \(\nu(X) = \phi(X)\), which is a valid measure if \(p < \infty\) and \(\phi\) is a pos. op. Use MCT.

4) via Radon-Nikodým, \(\int g \, d\phi(X) = \int f \, d\mu\).

5) use MCT, a few more steps. \(\square\)

If \(X\) is \(\sigma\)-finite, require absolute continuity, but not if \(\mu(X) < \infty\).

Hence we can talk about weak convergence.

- If \(1 \leq p < \infty\), \((f_n) \subseteq L^p\) converges weakly \(f_n \rightharpoonup f\) if

  \[ \forall g \in L^q : \left( \int |f_n - f| \, d\mu \right) \to 0. \]

- If \(p = \infty\), \(L^\infty = L^1\), \(\lim \int |f_n - f| \, d\mu = 0\), we only have weak-\(\ast\) convergence.

Example: Fix any \(g \in L^q, 1 \leq p < \infty\). The following \((f_n) \subseteq L^p\) converge weakly but not strongly to \(0\).

\begin{itemize}
  \item Oscillation: \(f_n(x) = g(x) \sin(nx)\)
  \item Concentration: \(f_n(x) = n^{1/p} g(nx)\)
  \item Escape to infinity: \(f_n(x) = g(x-n)\)
\end{itemize}

Recall \(f_n \rightharpoonup f\) if \(f_n \to f\) in \(L^1\).

Then Corollary of Banach-Alaoglu.

\((f_n) \subseteq L^p\) is bounded, \(1 \leq p < \infty\), then there is a weakly convergent subsequence.