Orthonormal Bases in Hilbert Space
APPM 5440/5450 Applied Analysis

Stephen Becker

Nov 17 2014, updated January 21, 2016

Abstract

Supplementary notes to our textbook (Hunter and Nachtergaele). These notes generally follow Kreyszig's book. The reason for these notes is that this is a simpler treatment that is easier to follow; the simplicity is because we generally do not consider uncountable nets, but rather only sequences (which are countable). I have tried to identify the theorems from both books; theorems/lemmas with numbers like 3.3-7 are from Kreyszig.

Note: there are two versions of these notes, with and without proofs. The version without proofs will be distributed to the class initially, and the version with proofs will be distributed later (after any potential homework assignments, since some of the proofs maybe assigned as homework).

1 Basic Definitions

NOTE: Kreyszig defines inner products to be linear in the first term, and conjugate linear in the second term, so $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle = \langle x, \bar{\alpha} y\rangle$. In contrast, Hunter and Nachtergaele define the inner product to be linear in the second term, and conjugate linear in the first term, so $\langle \bar{\alpha} x, y\rangle = \alpha\langle x, y\rangle = \langle x, \alpha y\rangle$. We will follow the convention of Hunter and Nachtergaele, so I have re-written the theorems from Kreyszig accordingly whenever the order is important. Of course when working with the real field, order is completely unimportant.

Let X be an inner product space. Then we can define a norm on X by

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad x \in X.$$

Thus, X is also a vector space (or normed linear space), and we can discuss completeness on X.

Definition 1.1. *A* ***Hilbert space****, typically denoted $\mathcal{H}$, is a complete inner product space. One must specify the field, and we will always assume it is either $\mathbb{R}$ or $\mathbb{C}$.*

Definition 1.2. *Let X and Y are inner product space over F. A mapping $T : X \to Y$ is an* ***isomorphism*** *if it is an* ***invertible*** *linear transformation from X* ***onto*** *Y such that*

$$\langle Tx, Ty\rangle = \langle x, y\rangle, \quad x, y \in X.$$

In this case, X and Y are said to be ***isomorphic****.*

The above isomorphism definition is in Hunter and Nachtergaele, but not until §8 (cf. Def. 8.28).

Lemma 1.3 (Lemma 3.3-7)**.** *Let M be a non-empty subset of a Hilbert space $\mathcal{H}$. Then the span(M) is dense in $\mathcal{H}$ if and only if $M^\perp = \{0\}$.*

Proof. Assume that $V = \text{span}(M)$ is dense in $\mathcal{H}$. Let $x \in M^\perp$. Since $\mathcal{H} = \overline{V}$ and $x \in \mathcal{H}$, there is a sequence $x_n \in V$ such that $x_n \to x$. For each n, we have $\langle x_n, x\rangle = 0$. By continuity of inner products,

$$0 = \langle x_n, x\rangle \to \langle x, x\rangle.$$

Thus, $x = 0$. Hence $M^\perp = \{0\}$.

Conversely, we assume $M^\perp = \{0\}$. Note that we have $\mathcal{H} = \overline{V} \oplus \overline{V}^\perp$. Let $x \in \overline{V}^\perp$. Then $x \perp V$. Consequently, we have $x \perp M$. Thus, by assumption, we have $x = 0$. That is $\overline{V}^\perp = \{0\}$. Hence $\mathcal{H} = \overline{V}$. □

2 Orthonormal sets

A first result is exercise 6.6 in Hunter/Nachtergaele (or lemma 3.4-2 in Kreyszig): vectors in an orthogonal set are linearly independent.

Theorem 2.1 (Thm. 3.4-6). *(Bessel inequality) Let (e_k) be an orthonormal sequence in an inner product space X. For every $x \in X$, we have*

$$\sum_{k=1}^{\infty} |\langle e_k, x\rangle|^2 \le \|x\|^2.$$

Proof. Let $x \in X$. We define y by

$$y = \sum_{k=1}^{n} \langle e_k, x\rangle e_k$$

and $z = x - y$. Notice that

$$\|y\|^2 = \sum_{k=1}^{n} |\langle e_k, x\rangle|^2 \quad \text{and} \quad \langle x, y\rangle = \sum_{k=1}^{n} \langle e_k, x\rangle\langle x, e_k\rangle = \|y\|^2$$

where the latter follows from linearity (since y is defined as a *finite* sum). So $\langle z, y\rangle = \langle x - y, y\rangle = \langle x, y\rangle - \langle y, y\rangle = \|y\|^2 - \|y\|^2 0$. Thus, we have $\|x\|^2 = \|y\|^2 + \|z\|^2$. So,

$$0 \le \|z\|^2 = \|x\|^2 - \|y\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle e_k, x\rangle|^2.$$

Hence

$$\sum_{k=1}^{n} |\langle e_k, x\rangle|^2 \le \|x\|^2.$$

Letting $n \to \infty$ yields the result. □

Definition 2.2. *Let (e_k) be an orthonormal sequence in an inner product space X. Let $x \in X$. The quantities $\langle e_k, x\rangle$ are called the **Fourier coefficients** of x with respect to the orthonormal sequence (e_k).*

Now we will discuss the convergence of the following **Fourier series**:

$$\sum_{k=1}^{\infty} \langle e_k, x\rangle e_k.$$

Theorem 2.3 (Thm. 3.5-2). *Let (e_k) be an orthonormal sequence in a Hilbert space $\mathcal{H}$. Then:*

(a) The series $\sum_{k=1}^{\infty} \alpha_k e_k$ converges if and only if the series $\sum_{k=1}^{\infty} |\alpha_k|^2$ converges.

(b) If the series $\sum_{k=1}^{\infty} \alpha_k e_k$ converges, we write $x = \sum_{k=1}^{\infty} \alpha_k e_k$, then we have $\alpha_k = \langle e_k, x \rangle$, and consequently,

$$x = \sum_{k=1}^{\infty} \langle e_k, x \rangle e_k.$$

(c) For any $x \in \mathcal{H}$, the series $\sum_{k=1}^{\infty} \langle e_k, x \rangle e_k$ converges. (Note that the sum may not equal to x.)

Proof. (a) Let $s_n = \alpha_1 e_1 + \cdots + \alpha_n e_n$ and $\sigma_n = |\alpha_1|^2 + \cdots + |\alpha_n|^2$. Then for any $n > m$,

$$\|s_n - s_m\|^2 = \|\alpha_{m+1} e_{m+1} + \cdots + \alpha_n e_n\|^2 = |\alpha_{m+1}|^2 + \cdots + |\alpha_n|^2 = \sigma_n - \sigma_m.$$

Thus s_n is Cauchy if and only if σ_n is Cauchy. This completes the proof.

N.B. The proof also follows from exercise 1.20 in Hunter/Nachtergaele: in a normed space X, every absolutely convergent sequence converges iff X is Banach. The statement here is simpler, since we do not need the *iff*, and also because due to orthogonality, we have an equality in the above equation (without orthogonality, we would resort to the triangle inequality and have an inequality instead of an equality).

(b) Let k be fixed. Then for any $n \geq k$, we have $\alpha_k = \langle e_k, s_n \rangle$. Since $s_n \to x$, and the inner product is (sequentially) continuous, we have $\alpha_k = \langle e_k, x \rangle$.

(c) By Bessel inequality, the sequence $\sum_{k=1}^{n} |\langle e_k, x \rangle|^2$ is increasing and bounded, and hence convergent. Then the result follows from (a).

□

In the above Theorem 2.3(c), we know that the sum may not be x itself. To ensure the sum is x, we need to assume that (e_k) is a **total** orthonormal set which is defined as follows.

Definition 2.4. *Let X be an inner product space and let M be a subset. Then M is called* ***total*** *if $\overline{span(M)} = X$.*

Definition 2.5. *A total orthonormal set in an inner product space is called an* ***orthonormal basis***. ***N.B.*** *Other authors, such as Reed and Simon, define an orthonormal basis as a* maximal *orthonormal set, e.g., an orthonormal set which is not properly contained in any other orthonormal set. The two definitions are equivalent (Hunter and Nachtergaele's theorem).*

Theorem 2.6 (Thm. 4.1-8)**.** *Every Hilbert space contains a total orthonormal set. (Furthermore, all total orthonormal sets in a Hilbert space $\mathcal{H} \neq \{0\}$ have the same cardinality, which is known as the* ***Hilbert dimension****).*

See Kreyzsig, where he states this without proof in §3.6 and proves it in §4.1. The corresponding result in Hunter/Nachtergaele is Theorem 6.29. The proof requires the axiom of choice or Zorn's lemma.

Proof. Let M be the set of all orthonormal subsets of $\mathcal{H}$. Pick any nonzero element x. Then one such orthonormal subset is $\{y\}$ where $y = x/\|x\|$, so $M \neq \emptyset$. Set inclusion defines a partial ordering on M, and every chain $C \subset M$ has an upper bound, namely the union of all subsets of X which are elements of C. By Zorn's lemma, M has a maximal element F. We claim F is total in $\mathcal{H}$. If not, then there is a nonzero $z \in \mathcal{H}$ such that $z \perp F$, but then we could have normalized and added z to F to create a larger orthonormal set, which is impossible by the maximality of F (similar to Lemma 1.3). □

Theorem 2.7 (Thm. 3.6-2)**.** *Let M be a subset of an inner product space X. Then:*

(a) If M is total in X, then $x \perp M$ implies $x = 0$.

(b) Assume that X is complete. If $x \perp M$ implies $x = 0$, then M is total in X.

Proof. (a) Let $x \perp M$. Since M is total, there is a sequence (x_n) in $\text{span}(M)$ such that $x_n \to x$. Note that $\langle x_n, x\rangle = 0$. By continuity of inner products, $0 = \langle x_n, x\rangle \to \langle x, x\rangle$. Thus, $x = 0$. (Note: this is essential Lemma 1.3, but that lemma assumed the space as complete, so we re-prove it here).

(b) This follows from Lemma 1.3.

□

Lemma 2.8 (Lemma 3.5-3)**.** *Let X be an inner product space and let $x \in X$. Let (e_k), $k \in I$, be an orthonormal set in X. Then at most countably many of the Fourier coefficients $\langle e_k, x\rangle$ are non-zero.* (This lemma is the key to Kreyszig's simpler approach to non-separable Hilbert spaces)

Proof. Let m be a positive integer. Suppose there are countably infinitely many $\langle e_k, x\rangle$ such that $|\langle e_k, x\rangle| > 1/m$. Notice that, for any n, we have

$$\sum_{k=1}^{n} |\langle e_k, x\rangle|^2 \leq \|x\|^2.$$

Thus, we have

$$\frac{n}{m^2} \leq \|x\|^2$$

for all n. This is a contradiction. Hence, there are only finitely many k with $|\langle x, e_k\rangle| > 1/m$. This proves the lemma. □

Let (e_k), $k \in I$, be an orthonormal set in an inner product space X. Let $x \in X$. The above lemma shows that the sum $\sum_k |\langle e_k, x\rangle|^2$ is a countable sum. Hence, Bessel's inequality can be applied to conclude $\sum_k |\langle e_k, x\rangle|^2 \leq \|x\|^2$ in this case.

Now we have the following criterion for totality.

Theorem 2.9 (Thm. 3.6-3)**.** *Let M be an orthonormal set in a Hilbert space $\mathcal{H}$. Then M is total in $\mathcal{H}$ if and only if for all $x \in \mathcal{H}$, the following **Parseval relation** holds (where we are summing over the non-zero terms only)*

$$\sum_k |\langle e_k, x\rangle|^2 = \|x\|^2.$$

Proof. Assume that $\sum_k |\langle e_k, x\rangle|^2 = \|x\|^2$ for all $x \in \mathcal{H}$. Suppose M is not total. Then $M^\perp$ contains a non-zero element, say $y \in M^\perp$ and $y \neq 0$. Then $\langle e_k, y\rangle = 0$ for each e_k in M. By assumption, we have $\|y\| = 0$. This is a contradiction. Hence M is total.

Conversely, we assume M is total. Let $x \in \mathcal{H}$. Let (e_k) be the sequence of elements in M such that $\langle e_k, x\rangle$ is nonzero (i.e., assume M is countable, or if not, that we are only indexing the ones that have nonzero coefficients). We define

$$y = \sum_{k=1}^{\infty} \langle e_k, x\rangle e_k$$

which is convergent since $x \in \mathcal{H}$. We will show $x - y \perp M$. Note that, for each e_j, we have

$$\langle e_j, x - y\rangle = \langle e_j, x\rangle - \sum_{k=1}^{\infty} \langle e_k, x\rangle \langle e_j, e_k\rangle = 0.$$

For any element $v \in M$ that is not equal to e_k for all k, we have $\langle v, x\rangle = 0$ since M is an orthonormal set. Thus,

$$\langle v, x - y\rangle = \langle v, x\rangle - \sum_{k=1}^{\infty} \langle e_k, x\rangle \langle v, e_k\rangle = 0.$$

Combining results, we have $x - y \in M^\perp$. Since $M^\perp = \{0\}$, we have $x = y$, that is

$$x = \sum_{k=1}^{\infty} \langle e_k, x \rangle e_k.$$

By sequential continuity of the inner product,

$$\|x\|^2 = \langle x, x \rangle = \left\langle x, \sum_{k=1}^{\infty} \langle e_k, x \rangle e_k \right\rangle = \sum_{k=1}^{\infty} \langle e_k, x \rangle \langle x, e_k \rangle = \sum_k |\langle e_k, x \rangle|^2.$$

□

Now we discuss Hilbert spaces that contain countable orthonormal sets. Note: the following theorem is the same as exercise 6.10 in Hunter/Nachtergaele ("A Hilbert space is a separable metric space iff it has a countable orthonormal basis"):

Theorem 2.10 (Thm. 3.6-4)**.** *Let $\mathcal{H}$ be a Hilbert space.*

1. *If $\mathcal{H}$ is separable, every orthonormal set in H is countable.*
2. *If $\mathcal{H}$ contains an orthonormal sequence which is total in $\mathcal{H}$, then $\mathcal{H}$ is separable.*

Proof.

1. Let B be a countable dense subset of $\mathcal{H}$. Let M be an uncountable orthonormal set. Then the distance between any two elements $x, y \in M$ is $\sqrt{2}$ since

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 = 2.$$

For each $x \in M$, we define N_x as the ball centered at x with radius $\sqrt{2}/3$. Then the N_x are disjoint. Since B is dense, for each x, there is $b_x \in B$ such that $b_x \in N_x$. Since N_x are disjoint, the collection $\{b_x\}$ is an uncountable subset of B. This is a contradiction.

2. Let (e_k) be a total orthonormal sequence in $\mathcal{H}$. It is true that then $\mathrm{span}((e_k))$ is dense in $\mathcal{H}$, but the span of a countable set is uncountable, since the field ($\mathbb{R}$ or $\mathbb{C}$) is uncountable. Instead, we do the following trick which is quite useful.

Let A be the set of all linear combinations

$$\gamma_1^{(n)} e_1 + \ldots + \gamma_n^{(n)}, \quad n = 1, 2, \ldots$$

where the coefficients γ are complex rational ($\gamma = a + ib$, both a and b rational) or just rational (if the underlying field was real). Think of A as the *rational-span* of (e_k).

Then A is the countable union of countable sets. We claim it is dense in $\mathcal{H}$. Fix any $x \in \mathcal{H}$ and any $\epsilon > 0$. Then since the sequence (e_n) is total in $\mathcal{H}$, there is some point $y = \sum_{k=1}^{n} \langle e_k, x \rangle e_k$ such that $\|x - y\| < \epsilon/2$, and then se use the triangle inequality and the density of the rationals to see that there is a nearby point v such that $\|x - v\| < \epsilon$.

□

And the major result of Hilbert space theory is the following:

Theorem 2.11 (Thm. 3.6-5)**.** *Two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, over the same field ($\mathbb{R}$ or $\mathbb{C}$), are isomorphic if and only if they have the same (Hilbert) dimension.*

Proof. Let the spaces be isomorphic and T an isomorphism from $\mathcal{H}$ to $\mathcal{H}'$. From the definition of isomorphism, we see that the map preserves orthonormality. Since T is bijective, it also preserves totality, so we can map a total orthonormal set in $\mathcal{H}$ to one in $\mathcal{H}'$.

Conversely, suppose they have the same dimension (and are not just the spaces $\{0\}$). We can construct an isomorphism by identifying their orthonormal bases. See Kreyszig for the details. □

3 Advanced definitions

The following is taken from Combettes and Bauschke's "Convex Analysis and Monotone Theory in Hilbert Spaces". Let M be a nonempty set and let $\preceq$ a binary relation on $M \times M$. Consider the following statements, and for all statements, let a and b be any arbitrary element of M, then

1. $a \preceq a$.
2. $(\forall c \in M), \quad \big(a \preceq b \text{ and } b \preceq c\big) \implies a \preceq c$.
3. $(\exists c \in M), \quad a \preceq c \text{ and } b \preceq c$.
4. $\big(a \preceq b \text{ and } b \preceq a\big) \implies a = b$.
5. $a \preceq b$ or $b \preceq a$.

if (1), (2) and (3) hold, then we call $(M, \preceq)$ a **directed set**. If (1), (2) and (4), we call $(M, \preceq)$ a **partially ordered set**. A partially ordered set with the property (5) is a **totally ordered set** (or a **chain**).

This allows us to define Zorn's lemma, and to talk about nets.

3.1 Nets

From §1.4 in Combettes/Bauschke, who follow §2 in Kelley's classic topology text "General Topology" (1955). The idea is to generalize the notion of a sequence, so that in this generalized notion, continuous is always the same as sequentially continuous (with the definition of sequentially continuous suitably modified) in all topologies. The theory of nets is due to Moore and Smith (1922); a similar theory that generalizes sequences, using the notion of a **filter**, is due to Cartan in 1937. Neither theory will be relevant for our class.

Let $(\mathcal{A}, \preceq)$ be a directed set. We write $b \succeq a$ to mean $a \preceq b$. Let X be a nonempty set. A **net** or **generalized sequence** in X indexed by $\mathcal{A}$ is denoted by $(x_a)_{a \in \mathcal{A}}$ (or just (x_a) if $\mathcal{A}$ is clear from context). For example, taking $(\mathbb{N}, \leq)$, we see that every sequence is a net. Not every net is a sequence, since we may have an uncountable index set $\mathcal{A}$.

Let (x_a) be a net in X, and let $Y \subset X$. We say that (x_a) is **eventually** in Y if there is some $c \in \mathcal{A}$ such that for all $a \in \mathcal{A}$, $a \succeq c \implies x_a \in Y$. We say the net is **frequently** in Y if for all $c \in \mathcal{A}$, there is some $a \in \mathcal{A}$ such that $a \succeq c$ and $x_a \in Y$.

3.2 Zorn's lemma

We follow Kreyszig again.

Definition 3.1 (Chain)**.** *A **chain**, or **totally ordered set**, is a partially ordered set such that every two elements are comparable.*

An **upper bound** of a subset W of a partially ordered set M is an element $u \in M$ such that $x \preceq u$ for every $x \in W$. Such an element need not exist. A **maximal element** of M is some $m \in M$ such that $m \preceq x$ implies $m = x$. Again, such an element need not exist, and if it does, it need not be an upper bound.

Note the funny definition of a maximal element, which is not the same as defining a **greatest element** (an element m such that $x \preceq m$ for all $x \in W$; i.e., an *upper bound* that lives within the set). These notions are not the same, since for a maximal element, we may not be able to compare $x \preceq m$ (or $m \preceq x$) for some x. If the set is totally ordered, then maximal and greatest elements are the same concept.

Similarly we can define a **least element**. A **well-ordered set** is a totally ordered set with the property that every subset has a least element. Zermelo's theorem says that every set can be well-ordered; that is, we can define some $\preceq$ to make it a well-ordered set. This is equivalent to the axiom of choice, and highly counter-intuitive.

Here are three common partial orderings:

1. the usual ordering $\leq$, on subsets of the real numbers (e.g., arbitrary subsets, or integers or $\mathbb{Q}$ or $\mathbb{N}$); this is also a total ordering. Note that $\mathbb{C}$ does not have a canonical total ordering.

2. The ordering on sets of sets, defined by set inclusion $\subset$. E.g., we say $A \preceq B$ if $A \subset B$. For example, if we have $A = \{1, 2\}$ and $B = \{2, 3\}$, then neither $A \subset B$ nor $B \subset A$, so we do not have a total ordering.

   As an example, let $M = \{\{a\}, \{b\}, \{a, b\}\}$, then $\{a, b\}$ is both an upper bound and a maximal element. If $M = \{\{a\}, \{b\}, \{c\}, \{a, b\}\}$, then there is no upper bound, and there are two maximal elements, namely $\{c\}$ and $\{a, b\}$.

3. The ordering induced by the positive semi-definite (PSD) cone. We work on the space of symmetric matrices, and we say $0 \preceq A$ if A is a PSD matrix, and we write $B \preceq A$ if $0 \preceq A - B$, i.e., if $A - B$ is PSD. Consider a matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, so then neither $A - B$ nor $B - A$ is PSD, so this is not a total ordering. However, it is a *directed* set, since for any two matrices A and B, we can alway find a third matrix C such that $C - A$ and $C - B$ are both PSD.

The following "lemma" implies, and is implied by, the axiom of choice, so it is often taken to be synonymous with it.

Lemma 3.2 (Zorn's lemma, 4.1-6)**.** *Let $M \neq \emptyset$ be a partially ordered set, and suppose every chain $C \subset M$ has an upper bound, then M has at least one maximal element.*

Theorem 3.3 (Hamel basis, 4.1-7)**.** *Every vector space $X \neq \{0\}$ has a Hamel basis.*

Proof. Let M be the set of all linearly independent subsets of X. Clearly this is non-empty. Define a partial ordering using set inclusion. Every chain $C \subset M$ has an upper bound, namely the union of all subsets of X which are elements of C. By Zorn's lemma, M has a maximal element B. This is a Hamel basis. Let $Y = \text{span}(B)$, so Y is a subspace of X, and $Y = X$ since otherwise we could add some element $z \in X \setminus Y$ to B, which would contradict the maximality of B. □