Pretest (Selected Solutions)
APPM 5450 Spring 2016 Applied Analysis 2

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Instructions: Take this *quickly* without spending over 30 minutes. It will not be graded and is intended to help you find weaknesses from last semester.
These are mainly True/False questions:

1. A norm is a convex function True. Ch. 1
2. limsup is always well-defined True, though the value may be infinite. Ch. 1
3. If T is linear and $x_n \to x$, then $T(x_n) \to T(x)$ False: this is true if T is sequentially continuous, i.e., continuous, but this is true iff it is bounded. Linear operators in infinite dimensional spaces need not be bounded. e.g., the differential operator. Ch. 1
4. If f is continuous over a compact set K, then it is also uniformly continuous True, cf Thm 1.67. Ch. 1
5. If f is continuous and G is an open set in its domain, then $f(G)$ is open as well False: if f is continuous, then if G is open in its range, $f^{-1}(G)$ is open. Ch. 1
6. A separable space also compact False, e.g., $\mathbb{Q}$. Ch. 1
7. A compact space separable True, cf lemma 1.63. The idea is that the countable union of countable sets is countable. Ch. 1
8. If (f_n) is a sequence of continuous functions converging to f pointwise, then f is also continuous False, we need uniform convergence for this to be true. Ch. 2
9. If X is a metric space, then $C(X)$ is a Banach space False: it is not even a normed space, since the uniform norm — which is what is implied when we write C — may not even be finite. If we either require the functions to be bounded, as in $C_b(X)$, or require X to be compact, then this statement is true. Ch. 2
10. Polynomials are dense in $C([a, b])$ True. Ch. 2
11. $C([a, b])$ is separable True, follows by density of polynomials. Ch. 2
12. If K is compact, then a subset of $C(K)$ is pre-compact if it is bounded False. This is Arzela-Ascoli, and we need the additional assumption that the subset is equicontinuous. Ch. 2
13. Let X be a complete metric space and $T : X \to X$ satisfy $d(T(x), T(y)) < d(x, y)$ for all $x, y \in X$, then $\exists!\, x$ such that $x = Tx$ False: this is almost the contraction mapping theorem, but that theorem requires that $d(T(x), T(y)) \leq c \cdot d(x, y)$ for some $c < 1$, which is not the same statement (make sure you agree with this). Ch. 3
14. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X, then the identity mapping $\mathcal{I} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous False: this is true iff $\mathcal{T}_1$ is finer than $\mathcal{T}_2$. Ch. 4
15. For which p is $\ell^p(\mathbb{N})$ a normed linear space? Banach space? Hilbert space? It is normed (and Banach) iff $1 \leq p \leq \infty$, and Hilbert iff $p = 2$. Ch. 5
16. $C^k([a, b])$ is a Banach space with the uniform norm? False, unless $k = 1$. Ch. 5
17. A subspace is necessarily a closed set False, but true in finite dimensions. Ch. 5

18. A Hamel basis is such that every element of the space can be written as a finite linear combination of basis elements True. Ch. 5

19. Every Banach space has a Schauder basis False, and still false even if the space is separable. Ch. 5

20. If X and Y are normed linear spaces and M is a dense subspace of X, and $T \in \mathcal{B}(M, Y)$, then there is a unique extension of T to all of X False, since we require that Y is Banach. Ch. 5

21. Let $T \in \mathcal{B}(X, Y)$ for Banach spaces X and Y, then T^{-1} is bounded as well False: we require that T is bijective. This is the open mapping theorem. Ch. 5

22. The right-shift operator S on ℓ^∞ is one-to-one True, as it has only a trivial kernel. Ch. 5

23. The right-shift operator S on ℓ^∞ is onto False. Ch. 5

24. Let X and Y be Banach and $T \in \mathcal{B}(X, Y)$, then if the kernel of T is trivial, there exists $c > 0$ such that $\forall x$, $\|Tx\| \geq c\|x\|$ False, we require that T has closed range (which is always true if it is finite rank). This is basically the open mapping theorem. Ch. 5

25. Let X be a normed linear space, then its dual $X^* = \mathcal{B}(X, \mathbb{R})$ a Banach space. True, cf. Thm 5.41. Ch. 5

26. $x_n \rightharpoonup x$ means $\forall \varphi \in X^*$, $\varphi(x_n) \to \varphi(x)$ True, this is the definition of weak convergence. Ch. 5

27. Let $(T_n) \subset \mathcal{B}(X, Y)$, then if $\|T_n - T\| \to 0$, we say (T_n) converges *strongly* False. This is strong in the sense that it is not weak, but for operators we have special terminology. This convergence is called *uniform* convergence, while strong (operator) convergence is equivalent to pointwise convergence. Ch. 5

28. An operator $T \in \mathcal{B}(X, Y)$ is called *compact* if it maps bounded sets $B \subset X$ to compact sets $T(B) \subset Y$ False, because $T(B)$ need not be closed: we only require that it is precompact. Ch. 5

29. A compact operator has finite rank False. Finite rank operators are compact, but not necessarily vice-versa. Compact operators can be *approximated* by finite-rank operators. Ch. 5

30. In finite dimensions, weak convergence implies strong convergence True. Ch. 5

31. We say a sequence $(\varphi_n) \subset X^*$ converges to φ in the weak-* sense if it converges weakly with respect to X^{**}, that is, $\forall f \in X^{**}$, $f(\varphi_n) \to f(\varphi)$ False. This would just be called weak convergence in the dual space. It is equivalent to weak-* convergence iff the space is reflexive, e.g., $X^{**} = X$. Ch. 5

32. In a reflexive Banach space, the closed unit ball in X^* is weakly compact True, implied by the Banach-Alouglu theorem. Ch. 5