Homework 13
APPM 5450 Spring 2018 Applied Analysis 2

Due date: THIS HOMEWORK IS OPTIONAL (i.e., not graded) **Instructor**: Prof. Becker
Theme: Measurable theory, L^p spaces, convergence theorems

Reading You are responsible for reading the rest of chapter 12 in the the book, and skimming chapter 13.

Problem 1: Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued measurable functions on $\mathbb{R}$ such that $|f_n(x)| \le 1$ and $\lim_{n\to\infty} f_n(x) = 1$ for all x. Evaluate the following integral and make sure to justify your calculation:

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(\cos(x)) e^{-\frac{1}{2}(x-2\pi n)^2}\, dx.$$

For further practice on problems involving convergence theorems, please look at older exams for APPM5450, and also at older analysis prelims. You will see that most of these exams involve at least one question on limit theorems, reflecting their importance in the curriculum.

Problem 2:

a) Problem 12.8 from the book: if $f_n \to f$ pointwise almost everywhere, and $|f_n| \le g \in L^p$, prove $\lim_{n\to\infty} \|f_n - f\|_p = 0$ (Note: you should assume $1 \le p < \infty$. The result is not true for $p = \infty$. Can you find a counter-example?).

b) Counterpart to 12.8: find an example of $(f_n) \subset L^p$ s.t. $f_n \to f$ pointwise a.e., but the sequence does not converge in the L^p norm. Your counter-example need only be for one p.

Problem 3: Problem 12.13 from the book

a) Prove $L^\infty(X)$ is not separable (and note that this is independent of whether $\mu(X)$ is finite or infinite).

b) Prove $C([0,1])$ is not dense in $L^\infty([0,1])$.

Problem 4: Problem 12.17: prove the unit ball in $L^p([0,1])$ is not compact (i.e., in the strong topology).

Problem 5: Problem 12.18 from the book: find a bounded sequence in $L^1([0,1])$ that does not have a weakly convergent subsequence, and explain why this doesn't violate Banach-Alouglu (in either the form given in chapter 12, or in Theorem 5.61).

Problem 6: Take the final exam from the 2014 class. Simulate real test conditions: study for the test, then take the test in one-sitting, closed book. The test is at amath.colorado.edu/faculty/martinss/Teaching/APPM5450_2014s/final.pdf. You can find his solutions at amath.colorado.edu/faculty/martinss/Teaching/APPM5450_2014s/final_solns.pdf.

Problem 7: Read problems 12.14 and 12.15 from the book (attempt if you like)

Problem 8: Problem 12.16 from the book. Use the book's hint, and also recall the hint from exercise 11.10. Don't forget to provide the counter-example for L^∞.

Problem 9: Term-by-term integration.

a) Use the (complete) Lebesgue measure. Let $u_k : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and suppose $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a,b]$, and define its limit as $f(x)$. Prove that f is integrable on $[a,b]$ and that

$$\int_a^b f(x)\, dx = \sum_{k=1}^{\infty} \int_a^b u_k(x)\, dx.$$

b) Prove that a power series $f_N(x) = \sum_{n=1}^N c_n(x-a)^n, x \in \mathbb{R}$, converges uniformly to its limit inside any closed interval inside the interval of convergence. (Define the radius of convergence as the largest $r \geq 0$ such that $f_N(x)$ converges absolutely/unconditionally pointwise to its limit for all $x \in (a-r, a+r)$ and define the interval of convergence as the open set $(a-r, a+r)$. It may or may not converge conditionally at the end-points)

   Hint: use the Weierstrass M-Test. Recall the *Weierstrass M-test*: if (f_n) is a sequence of functions from a metric space X to $\mathbb{R}$, and $|f_n(x)| \leq M_n$ for all $x \in X$ and $n \in \mathbb{N}$, then $\sum_n f_n$ converges uniformly if $\sum_n M_n$ converges.

c) Rigorously prove the following Taylor series expansion for arctan

$$\forall x \in [0,1), \quad \arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

   Hint: Note that $d/dx \arctan(x) = 1/(1+x^2)$, and find the Taylor series of $1/(1+x^2)$ using, e.g., the Neumann series.

Fact Term-by-term differentiation. (No work required). Suppose $f_n : [a,b] \to \mathbb{R}$ for $n \in \mathbb{N}$ has a *derivative* on $[a,b]$ (at a and b, we mean one-sided derivative), and $f_n \to f$ pointwise. Our question is: under what conditions do we have that $f_n' \to f'$ as well? This is not always true; for example (from §5.4.3 in Hunter's undergrad real analysis text www.math.ucdavis.edu/~hunter/m125a/intro_analysis.pdf), define

$$f_n(x) = \frac{x}{1+nx^2}$$

and $f_n \to f = 0$ uniformly, so $f' = 0$, but f_n' converges pointwise to the discontinuous function $\varphi \neq f'$ defined by $\varphi(0) = 1$ and $\varphi(x) = 0$ otherwise.

The appropriate theorem is that if if $f_n' \to \varphi$ *uniformly*, for some function φ, then $f' = \varphi$ (and hence f is differentiable, and in fact the f_n must have converged to f *uniformly*). See also corollary 12.36 in our book for other sufficient conditions.

For a power *series*, which is a very special type of sequence, we have stronger results. In particular, inside the radius of convergence of a Taylor series of a function f, the function f is C^∞ and its derivative is given by the series of derivatives.

See `https://www.dpmms.cam.ac.uk/~agk22/uniform.pdf` for example, as well as chapter 6 of Hunter's real analysis book.