Homework 12
APPM 5450 Spring 2018 Applied Analysis 2

Due date: Friday, April 27 2018, before 1 PM **Instructor**: Prof. Becker
Theme: Measure theory

Instructions Problems marked with "**Collaboration Allowed**" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.

On problems marked "**No Collaboration**," collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.

An arbitrary subset of these questions will be graded.

Reading You are responsible for reading section 12.3–12.6 in the book.

Problem 1: No Collaboration Problem 12.4 from the text: Give an example of a monotonic *decreasing* sequence of non-negative functions converging pointwise to f such that the result of the MCT does not hold.

Problem 2: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real-valued measurable functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = x$ for all $x \in \mathbb{R}$, i.e., f_n converges pointwise to the identity. Specify which of the following limits necessarily exist, and give a formula for the limit in the cases where this is possible (which may or may not depend on the exact sequence (f_n)).

a) **Collaboration Allowed**

$$\lim_{n\to\infty} \int_1^2 \frac{f_n(x)}{1+f_n(x)^2}\, dx$$

b) **No Collaboration**

$$\lim_{n\to\infty} \int_0^1 \frac{\sin(f_n(x))}{f_n(x)}\, dx, \quad \text{where we define} \quad \sin(0)/0 = 1$$

c) **No Collaboration**

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(f_n(x))}{f_n(x)}\, dx, \quad \text{where we define} \quad \sin(0)/0 = 1$$

d) **Collaboration Allowed**

$$\lim_{N\to\infty} \int_0^1 \sum_{n=1}^N \frac{|f_n(x)|}{n^2(1+|f_n(x)|)}\, dx$$

e) **No Collaboration**

$$\lim_{N\to\infty} \int_0^\infty \sum_{n=1}^N \frac{1}{n^2(1+|f_n(x)|^2)}\, dx$$

Problem 3: **Collaboration Allowed** Egorov's Theorem. Let (f_k) be a sequence of measurable functions that converges almost everywhere on E to a finite limit f, and $\mu(E) < \infty$. Use the (complete) Lebesgue measure. Prove:

a) Lemma. Given $\delta, \epsilon > 0$, prove there is a closed subset $F \subset E$ and an integer K such that $\mu(E \setminus F) < \epsilon$ and $|f(x) - f_k(x)| < \delta$ for all $k > K$ and $x \in F$. Hint: consider sets of the form $E_K = \{x \mid \forall k \geq K,\ |f(x) - f_k(x)| < \delta\}$.

b) Egorov's Theorem. Given $\epsilon > 0$, prove there is a closed subset $F \subset E$ such that $\mu(E \setminus F) < \epsilon$ and (f_k) converges uniformly to f on F. This remarkable result says that, under assumptions that the limit is finite and that $\mu(E)$ is bounded, then pointwise convergence actually implies uniform convergence except on sets of arbitrarily small measure (though not zero measure).

Problem 4: **Collaboration Allowed** Convergence in measure. Let (f_n) be measurable functions which are finite almost-everywhere on a set X. Then we say (f_n) *converges in measure* on X to f if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mu\left(\{x \in X \mid |f(x) - f_n(x)| > \epsilon\}\right) = 0.$$

We write this as $f_n \overset{m}{\to} f$.

a) Let (f_n) be measurable functions which are finite almost-everywhere on a set X. Suppose f_n converges pointwise a.e. to f, and $\mu(X) < \infty$. Prove $f_n \overset{m}{\to} f$. Note: in this problem, we mean convergence pointwise within the topology on $\mathbb{R}$, not $\overline{\mathbb{R}}$, so we would *not* say that $f_n(x) = n$ converges to $f(x) = \infty$.

b) Suppose $\mu(X) < \infty$, does $f_n \overset{m}{\to} f$ imply f_n converges to f pointwise almost-everywhere? Prove it does, or provide a counter-example that it doesn't.

Probability theory facts (no work required).

A **probability** space on an underlying set Ω (this is X in our more general notation) is just a measure μ (often labeled p or P in this context) and $\sigma-$algebra such that $\mu(\Omega) = 1$. Measurable sets correspond to *events* and measurable functions correspond to *random variables*. For example, the expected value of a random variable f is just $\mathbb{E}[f] \overset{\text{def}}{=} \int_\Omega f d\mu$. There are many forms of convergence of (f_n) (see, e.g., Definition 11.49 in the text for *convergence in distribution*, i.e., weak convergence.). Some forms of convergence imply the others. See basic probability books or http://en.wikipedia.org/wiki/Convergence_of_random_variables.

- $f_n \overset{m}{\to} f$ is called **convergence in probability** and often written $\boxed{f_n \overset{P}{\to} f}$.
- Another example (cf. Wasserman's "All of Statistics" book) is **convergence almost surely**, written $\boxed{f_n \overset{as}{\to} f}$, to mean the probability of convergence is 1, so this coincides with our notion of convergence **pointwise almost everywhere**. Specifically, it means $\mathbb{P}(\{x \mid f_n(x) \to f(x)\}) = 1$.
- We say f_n converges **in quadratic mean** to f if $\mathbb{E}\,|f_n - f|^2 \to 0$, written as $\boxed{f_n \overset{qm}{\to} f}$. Convergence in L^1 is written as $\boxed{f_n \overset{L^1}{\to} f}$, i.e., $\lim_{n\to\infty} \mathbb{E}\,|f_n - f| = 0$.
- Convergence **in distribution** is written $\boxed{f_n \rightsquigarrow f}$. It means, where F_t is the CDF of f_t, that $\lim_{n\to\infty} F_n(t) = F(t)$ for all t at which F is continuous. See Def. 11.49 in Hunter and Nachtergaele; for absolutely continuous r.v., this is equivalent to weak convergence.
- Theorems: (from Wasserman, §5.2 and §5.7; a–b from Thm. 5.4m, c–e from Thm. 5.17)
  a) $f_n \overset{qm}{\to} f$ implies $f_n \overset{P}{\to} f$, but not vice-versa
  b) $f_n \overset{P}{\to} f$ implies $f_n \rightsquigarrow f$, but not vice-versa
  c) $f_n \overset{as}{\to} f$ implies $f_n \overset{P}{\to} f$ (this is Problem 4(a); the lack of a converse is Problem 4(b)).

d) $f_n \xrightarrow{L^1} f$ implies $f_n \xrightarrow{P} f$

e) $f_n \xrightarrow{qm} f$ implies $f_n \xrightarrow{L^1} f$

f) $f_n \xrightarrow{P} b$ does not imply $\mathbb{E}(f_n) \to b$, e.g., let $f_n = n^2$ wp $1/n$ and 0 otherwise, so $\mathbb{P}(|f_n| < \epsilon) \to 0$ so $f_n \xrightarrow{P} 0$, but $\mathbb{E}(f_n) = n \to \infty$.

Hence $f_n \xrightarrow{qm} f$ is a strong statement, and $f_n \rightsquigarrow f$ is a rather weak statement.

- We say a *measure* μ is **absolutely continuous** (with respect to Lebesgue λ, written $\mu \ll \lambda$) if there is a function $p \in L^1(\mathbb{R})$ such that

$$\mu(A) = \int p(x)\, d\lambda$$

for every measurable set A. Note that $\mu(A) = \int 1\, d\mu$ by definition. In other words, "it has a pdf", which means that the pdf is described by a normal function $p \in L^1$, not a distribution. In contrast, a discrete distribution does "not have a pdf" which means that, with respect to Lebesgue, the pdf is not described by a normal function (rather, it needs to use a distribution like the delta function).

- Informally, not using measure theory notation, we say a random variable X is **continuous** if the probability X belongs to any singleton set is zero (in contrast to discrete random variables). This really relies on the underlying measure space, not just the measurable function. It is **absolutely continuous** if every set of Lebesgue measure zero has zero probability. Absolute continuity implies continuity, but not vice-versa, as one might expect from the terminology (continuous but not absolutely continuous are rare, and are called "singularly continuous random variables", like the Cantor function/Devil's staircase). See also §11.12 in our book for a brief discussion, and see also our handout "Absolutely continuous functions, Radon-Nikodym Derivative".

- Warning: informally, people may write "random variable" to refer to all of a probability space, and not really mean a measurable function on a probability space. Also, a "probability distribution" usually refers to a probability *space*, and not the notion of "distribution" we discussed in the Fourier Transform chapter.

- An absolutely continuous distribution need not have finite moments, e.g., the Cauchy distribution, given by $p(x) = \frac{1}{\pi(1+x^2)}$, has no finite moments of order great than one. In particular, the mean and variance are undefined! That is, $\mathbb{E}[X]$ is not finite!