Homework 11
APPM 5450 Spring 2018 Applied Analysis 2

Due date: Friday, April 20 2018, before 1 PM **Instructor**: Prof. Becker
Theme: Measurable theory

Instructions Problems marked with "**Collaboration Allowed**" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.

On problems marked "**No Collaboration**," collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.

An arbitrary subset of these questions will be graded.

Reading You are responsible for reading section 12.2 in the book.

Problem 1: **No Collaboration** By definition, a measure on a σ-algebra $\mathcal{A}$ is σ-additive (aka countably-additive), meaning that if $\{A_i\} \subset \mathcal{A}$ are pairwise disjoint, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. Note that it is easy to prove a measure is also **monotone**, i.e., $A \subset B \implies \mu(A) \leq \mu(B)$. Prove that a measure is also

a) **subadditive**, i.e., for any $N < \infty$, and $A_i \in \mathcal{A}$ *not* necessarily pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{N} A_i\right) \leq \sum_{i=1}^{N} \mu(A_i).$$

b) σ**-subadditive** (or countably subadditive), for $A_i \in \mathcal{A}$ *not* necessarily pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

Note that we are *not* subadditive for *uncountable* unions. For example, with the Lebesgue measure λ, we have $\lambda([0,1]) = 1$ but $[0,1] = \bigcup_{x\in[0,1]} M_x$ where $M_x = \{x\}$ so $\lambda(M_x) = 0$.

Problem 2: **Collaboration Allowed** Let $(X, \mathcal{A}, \mu)$ be a complete measurable space and consider the space $L^\infty = L^\infty(X, \mathcal{A}, \mu)$ consisting of all measurable functions from X to $\mathbb{R}$ such that

$$\|f\|_\infty = \text{ess sup}_{x\in X}|f(x)| < \infty.$$

Prove that L^∞ is closed under the $\|\cdot\|_\infty$ norm (i.e., that this is a Banach space).

Hint: You may want to follow this strategy:

a) Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in L^∞
b) For each positive integer k, there is N_k such that $m, n \geq N_k$ implies $\|f_n - f_m\|_\infty < 1/k$.
c) For each k, and for each $m, n \geq N_k$, let $\Omega^k_{m,n}$ be the set of all $x \in X$ such that $|f_m(x) - f_n(x)| < 1/k$. What can you say about $\Omega^k_{m,n}$ in light of (b)?
d) Set $\Omega^k = \bigcap_{m,n=N_k}^{\infty} \Omega^k_{m,n}$. What do you know about Ω^k in view of your conclusion from (c)?
e) Set $\Omega = \bigcap_{k=1}^{\infty} \Omega^k$. What do you know about Ω in view of your conclusion from (d)?

f) What can you say about $(f_n(x))_{n\in\mathbb{N}}$ for $x \in \Omega$?

It's instructive to first think of just $X = [a, b] \subset \mathbb{R}$ and show that the space of continuous functions is closed under the uniform norm (so we can use "sup" instead of "ess sup" in the definition of the norm). The idea is that if we have a Cauchy sequence, then for each point x, $f_n(x)$ is Cauchy, and a real number so it converges, and therefore we have a pointwise limit. Furthermore, we converge uniformly to this limit, so the limit also has bounded norm. See the proof of Theorem 2.4 in the book (it interchanges "sup" and "lim" using example 1.25, but you can also use the fact that we have a norm, so it is sequentially continuous).

Now, this problem is asking us to do this in the case when we may have uncontrolled behavior on sets of measure zero. We want to make sure that when we add together a lot of sets of measure zero, we don't get anything bad.

Problem 3: No Collaboration

Let μ be a Borel measure on $\mathbb{R}^n$. Prove that if $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is finite a.e. (with respect to μ on $\mathbb{R}^n$) and measurable, so that $\varphi(f)$ is defined a.e. in $\mathbb{R}^n$, then $\varphi(f)$ is measurable (w.r.t. μ and the standard Borel measure on $\overline{\mathbb{R}}$) if f is.

Problem 4: Collaboration Allowed

Let f and g be measurable functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, using the standard Borel σ-algebra on $\overline{\mathbb{R}}$ and an arbitrary σ-algebra $\mathcal{A}$ on $\mathbb{R}^n$. Assume both f and g are finite almost-everywhere; note that since $\mathbb{R}^n$ itself is measurable, then $\{x \in \mathbb{R}^n \mid f(x) = -\infty\}$ is a measurable set. Prove the following:

(a) The set $G = \{x \in \mathbb{R}^n | f(x) > g(x)\}$ is measurable.

(b) For any real number λ, $f + \lambda$ and λf are measurable.

(c) $f + g$ is measurable.

(d) f^2 is measurable.

(e) fg is measurable. (Hint: use some of the above)

Problem 5: Collaboration Allowed

a) Fact (no proof needed): if T is a Lipschitz continuous transformation of $\mathbb{R}^n$ to itself, then T maps measurable sets into measurable sets.

b) Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a measurable function. If T is a nonsingular linear transformation of $\mathbb{R}^n$, show that $f \circ T$ is measurable. (Hint: if $E_1 = \{x|f(x) > a\}$ and $E_2 = \{x|f(Tx) > a\}$, show that $E_2 = T^{-1}E_1$.)