Homework 10
APPM 5450 Spring 2018 Applied Analysis 2

Due date: Friday, April 13 2018, before 1 PM   **Instructor**: Prof. Becker
Theme: Measurable sets

Instructions Problems marked with "**Collaboration Allowed**" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.

On problems marked "**No Collaboration**," collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.

An arbitrary subset of these questions will be graded.

Reading You are responsible for reading section 12.1 in the book.

Problem 1: **No Collaboration** Extension of Exercise 12.2 (part (d) is new): Let $\mathcal{A}$ be a σ-algebra of subsets of Ω, and suppose μ is a measure on $(\Omega, \mathcal{A})$. Prove the following properties:

(a) if A, $B \in \mathcal{A}$, then $A \backslash B \in \mathcal{A}$;

(b) if A, $B \in \mathcal{A}$, and $A \subset B$, then $\mu(A) \leq \mu(B)$;

(c) if A, $B \in \mathcal{A}$, then $\mu(A \cup B) \leq \mu(A) + \mu(B)$;

(d) if A, $B \in \mathcal{A}$, then $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$.

Problem 2: **Collaboration Allowed** Exercise 12.3: If (A_i) is an increasing sequence of measurable sets, meaning that

$$A_1 \subset A_2 \subset \ldots \subset A_i \subset A_{i+1} \subset \ldots,$$

then prove that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$

If (A_i) is a decreasing sequence of measurable sets, meaning that

$$A_1 \supset A_2 \supset \ldots \supset A_i \supset A_{i+1} \supset \ldots,$$

and $\mu(A_1) < \infty$, prove that

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$

Give a counterexample to show that this result is not necessarily true if $\mu(A_i)$ is infinite for every i.

Problem 3: **No Collaboration** If $(E_k)_{k\in\mathbb{N}}$ is a sequence of sets, define

$$\limsup E_k = \bigcap_{j=1}^{\infty}\left(\bigcup_{k=j}^{\infty} E_k\right), \quad \liminf E_k = \bigcup_{j=1}^{\infty}\left(\bigcap_{k=j}^{\infty} E_k\right).$$

Prove $\limsup E_k$ consists exactly of those points which belong to infinitely many E_k, and that $\liminf E_k$ consists exactly of those points which belong to all E_k from some k on. Conclude that $\liminf E_k \subset \limsup E_k$.