Homework 9
APPM 5450 Spring 2018 Applied Analysis 2

Due date: Friday, April 6 2018, before 1 PM **Instructor**: Prof. Becker
Theme: Distributions

Instructions Problems marked with "**Collaboration Allowed**" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.

On problems marked "**No Collaboration**," collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.

An arbitrary subset of these questions will be graded.

Reading You are responsible for reading sections 11.4–1.8 in the book as usual, and please read sections 11.9 – 11.12 of the book as well, as they will not be covered fully in lecture. These are especially important if you are interested in probability theory.

Problem 1: **No Collaboration** Problem 11.4: if $\varphi \in \mathcal{S}(\mathbb{R})$, prove $\varphi\delta' = \varphi(0)\delta' - \varphi'(0)\delta$.

Problem 2: **Collaboration Allowed** Problem 11.6: show that the distributional derivative of $\log|x| : \mathbb{R} \to \mathbb{R}$ is p.v.$1/x$.

Problem 3: **Collaboration Allowed** Problem 11.14 about the Airy equation

Problem 4: **No Collaboration** Problem 11.15

Problem 5: **Collaboration Allowed** Problem 11.18: prove that if $g \in L^2(\mathbb{R})$ satisfies $g(-x) = \overline{g(x)}$ (almost everywhere), then $\hat{g}$ is real-valued. Hint: there is more to this than just doing some integrals.

Problem 6: **No Collaboration** Compute the Fourier transforms of

a) $f(x) = \chi_{[-R,R]}(x)$ (that is, the indicator function of the interval $[-R, R]$)

b) $f(x) = e^{-a|x|}$ where $a > 0$.

by evaluating the formula

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} f(x)\, dx$$

since these are functions in L^1. For each function f, <u>what space does $\widehat{f}$ live in</u>? [Note: after simplifying as much as possible, you may use integral tables or a computer to evaluate the integrals if you wish].

Problem 7: **No Collaboration** Problem 11.19: give a counterexample to show that the Riemann-Lebesgue lemma does not hold for all functions in L^2: that is, find $f \in L^2(\mathbb{R})$ such that $\hat{f}$ is not continuous.

Here are some review questions that are useful for understanding chapter 11

1. What does it mean for $\varphi_n \to \varphi$ in $\mathcal{S}$
2. Make absolutely sure that you understand problems like 11.4 and 11.10a
3. Prove that if $\varphi_n \to \varphi$ in $\mathcal{S}$, then $x\varphi_n(x) \to x\varphi(x)$ and $\partial\varphi_n \to \partial\varphi$ in $\mathcal{S}$
4. Let T be a linear map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathbb{R}$. What does it mean for T to be continuous? Prove that if there exists a finite C and a finite N such that $|T(\varphi)| \leq C \sum_{|\alpha|,|\beta| \leq N} \|\varphi\|_{\alpha,\beta}$, then T is continuous.
5. Let $T \in \mathcal{S}^*(\mathbb{R}^d)$, and α a multi-index. How do we define $x^\alpha T$? Prove that this is a tempered distribution.
6. Prove that $n^2 \sin(nx) \to 0$ in $\mathcal{S}^*$.
7. Is the Schwartz space dense in $\mathcal{S}^*$?
8. Prove that $\sup_x |x^\beta \partial^\alpha \varphi(x)| < \infty \; \forall \alpha, \beta$ iff $\sup_x |(1+|x|^2)^{k/2} \partial^\alpha \varphi(x)| < \infty \; \forall \alpha, k$.
9. Assume $\int_{\mathbb{R}^d} |f|^2 < \infty$ and define T_f by $\langle T_f, \varphi \rangle = \int_{\mathbb{R}^d} f\varphi$ (i.e., a regular distribution). Prove that $T \in \mathcal{S}^*$.
10. Let H be the Heaviside function (so $H(x) = 1$ if $x \geq 0$ and 0 otherwise). Prove $H \in \mathcal{S}^*$, and calculate H'. Let H_R denote the function that is 1 when $x \in [0, R]$ and zero otherwise. Prove $H_R \to H$ in $\mathcal{S}^*$ as $R \to \infty$.
11. Let $\psi \in \mathcal{S}$ such that $\int \psi = 0$. Set $\varphi_n(x) = n\psi(nx)$. Does φ_n converge in S? Does φ_n (i.e., T_{φ_n}) converge in $\mathcal{S}^*$?
12. Prove that P.V.$(1/x)$ is a continuous functional on $\mathcal{S}$.
13. What is the distributional derivative of P.V.$(1/x)$?
14. Define $\widehat{T}$ for $T \in \mathcal{S}^*$. Prove that what you defined is a continuous map on $\mathcal{S}$.