Homework 8
APPM 5450 Spring 2018 Applied Analysis 2

Due date: Friday, Mar 23 2018, before 1 PM **Instructor**: Prof. Becker
Theme: Distributions

Instructions Problems marked with "**Collaboration Allowed**" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.

On problems marked "**No Collaboration**," collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.

An arbitrary subset of these questions will be graded.

Reading You are responsible for reading sections 11.1–1.3 in the book.

Problem 1: **No Collaboration** 11.2 from the book. You should assume that the family $\{p_n\}$ separates points. Can this metric be induced from a norm, i.e., is $p(x) = \sum_n 2^{-n} \frac{p_n(x)}{1+p_n(x)}$ a norm?

Problem 2: **No Collaboration** 11.7 from the book. This one does not need to be rigorous.

Problem 3: **Collaboration Allowed** 11.10 from the book. For part (a), to show a linear transform $T : \mathcal{S} \to \mathcal{S}$ is continuous, you only need to find an estimate of the form used on the last line of the proof of Prop. 11.25, i.e., for all multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$, find an integer $d < \infty$ and constants $c_{\alpha',\beta'}$ s.t.

$$\forall \varphi \in \mathcal{S}, \quad \|T\varphi\|_{\alpha,\beta} \leq \sum_{\substack{|\alpha'| \leq |\alpha|+d \\ |\beta'| \leq |\beta|+d}} c_{\alpha',\beta'} \|\varphi\|_{\alpha',\beta'} \tag{1}$$

which is an extension of what we use to find continuous *functionals* (the second equation of §11.2). Also for part (a), you may prove the result for $\alpha = 0$ and arbitrary β, then prove it for arbitrary α and $\beta = 0$, and then claim (without having to show details) that it holds for arbitrary α and arbitrary β. For part (b), there is a tempting shortcut proof using sequential continuity, but it has a circular reasoning flaw.

Problem 4: **Collaboration Allowed**. Let k be a positive integer. Prove that there exist numbers c_k and C_k such that $0 < c_k \leq C_k < \infty$ and

$$c_k(1 + |x|^k) \leq (1 + |x|^2)^{k/2} \leq C_k(1 + |x|^k), \quad \forall x \in \mathbb{R}^d. \tag{2}$$

Check to see if you can readily adapt your proof to also prove the existence of numbers $0 < b_k \leq B_k < \infty$ such that

$$b_k(1 + |x|)^k \leq (1 + |x|^2)^{k/2} \leq B_k(1 + |x|)^k, \quad \forall x \in \mathbb{R}^d. \tag{3}$$

Hint: for given k and d, you could bound the numbers c_k and C_k explicitly, but it may be simpler to not find explicit bounds and instead use the properties of continuous functions.

Note 1 The book defined the family of seminorms

$$p_{\alpha,\beta}(\varphi) = \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta \varphi(x)|.$$

We can also consider another countable family of seminorms

$$q_{k,\beta}(\varphi) = \sup_{x\in\mathbb{R}^d} (1+|x|^2)^{k/2} |\partial^\beta \varphi(x)|$$

where k is a positive integer (rather than a multi-index). These two families of seminorms generate the same topology (on the set of C^∞ functions).

Note 2 The existence of inequalities such as (2) and (3) is routinely used (generally without commenting on it) to replace the growth factor $(1+|x|^2)^{k/2}$ in the seminorms $q_{k,\beta}$ by either $(1+|x|^k)$ or $(1+|x|)^k$, whenever convenient.

You may find it interesting to use a computer and see the behavior of the constants b_k, B_k, c_k, C_k as k grows large.