Homework 7
APPM 5450 Spring 2018 Applied Analysis 2

Due date: Monday, Mar 19 2018, before 1 PM **Instructor**: Prof. Becker
Theme: spectral theory

Instructions Problems marked with "**Collaboration Allowed**" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.

On problems marked "**No Collaboration**," collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.

An arbitrary subset of these questions will be graded.

Special comments The due-date is delayed due to midterms

Note In the spectral theorem, we use P_n as orthogonal projections onto a (finite-dimensional) eigenspace. We allow these eigenspaces to be more than 1 dimension so that we can group eigenvectors that have the same eigenvalue together. However, it is often more convenient to allow duplicate eigenvalues but require P_n to correspond to a single eigenvector. In that case, we still have a countable sum, and can write $P_n(x) = \langle e_n, x\rangle e_n$ for some orthonormal set (e_n) (and we can extend e_n to an orthonormal basis, where all the new additions are in the kernel — note that the spectral theorem implies that the orthogonal complement of the kernel of a compact operator is separable).

That is, if A is compact and self-adjoint on $\mathcal{H}$,

$$A = \sum_{n=1}^{\infty} \lambda_n \langle e_n, \cdot\rangle e_n, \quad e_n \text{ orthonormal eigenvectors}$$

With this notation, the connections with Bessel's inequality and Parseval's equality should be more clear.

Problem 1: **No Collaboration** 9.20 from the book: let $A = A^* \in \mathcal{B}(\mathcal{H})$ be compact, and f a complex valued continuous function defined on $\sigma(A)$. When is $f(A)$ compact? Make sure to provide an "if-and-only-if" characterization.

Problem 2: **Collaboration Allowed** 9.22 from the book. Suppose $A \in \mathcal{B}(\mathcal{H})$ is compact and non-negative. Prove there is a unique compact nonnegative linear operator B such that $B^2 = A$ (and so we write $B = A^{1/2}$). [Note: since A is a nonnegative operator, using the definition from §8, you may assume it is self-adjoint.]

Problem 3: **Collaboration Allowed** Consider the Hilbert space $\mathcal{H} = \mathbb{C}^n$ and $A \in \mathcal{B}(\mathcal{H})$, with $(e^{(j)})_{j=1}^n$ the canonical basis, and the representation of A with respect to the canonical basis being

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Define the Hilbert-Schmidt (aka Frobenius) norm of A by

$$\|A\|_{\text{HS}} = \left(\sum_{i,j=1}^{n} |a_{i,j}|^2 \right)^{1/2}$$

a) Let $(\varphi^{(j)})_{j=1}^{n}$ be any orthonormal basis for $\mathcal{H}$. Show that $\|A\|_{\text{HS}}^2 = \sum_{j=1}^{n} \|A\varphi^{(j)}\|^2$. (Note: do not use the facts from the book, e.g., problem 9.12, unless you prove those)

b) Show $\|A\| \leq \|A\|_{\text{HS}} \leq \sqrt{n}\|A\|$ for all $A \in \mathcal{B}(\mathcal{H})$.

c) Find examples of $A \in \mathcal{B}(\mathcal{H})$ to show that both inequalities above are tight.