Homework 6
APPM 5450 Spring 2018 Applied Analysis 2

Due date: Friday, Mar 9 2018, before 1 PM **Instructor**: Prof. Becker
Theme: spectral theory

Instructions Problems marked with "**Collaboration Allowed**" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.

On problems marked "**No Collaboration**," collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.

An arbitrary subset of these questions will be graded.

Reading You are responsible for reading section 9.5 in the book (and 9.4 if you didn't read it last week)

Problem 1: **Collaboration Allowed** Problem 9.6 about $Gf(x) = g(x)f(x)$ for some $g \in C_b(\mathbb{R})$. You may assume g is real-valued. Note that the overline notation means closure, as is usual, and not the conjugate set as in problem 9.1. Please do this problem rigorously. Prove (a) $G \in \mathcal{B}(L^2(\mathbb{R}))$, (b) $\sigma(G) = \overline{\{g(x) \mid x \in \mathbb{R}\}}$, and (c) answer if/when an operator of this form can have eigenvalues.

Problem 2: **Collaboration Allowed** Problem 9.9.
Hint (also, see the book for a hint):
use the Fourier domain.

Problem 3: **No Collaboration** Problem 9.11 (From the statement proved here, a very important fact follows: if $\lambda \in \sigma_c(A)$, then there exists a sequence of vectors (x_n) with $\|x_n\| = 1$ such that $\|(A - \lambda I)x_n\| \to 0$.)

Problem 4: **No Collaboration** Problem 9.19. Let $A = A^* \in \mathcal{B}(\mathcal{H})$ be a compact self-adjoint linear operator with where $A = \sum_{n=1}^{\infty} \lambda_n P_n$. We define $f(A) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} f(\lambda_n) P_n$. Prove that the series in this definition converges in the strong operator topology if f is continuous (on $\sigma(A)$), and converges uniformly if additionally $f(0) = 0$.