Homework 5
APPM 5450 Spring 2018 Applied Analysis 2

Due date: Friday, Mar 2 2018, before 1 PM **Instructor**: Prof. Becker
Theme: Basic spectral theory

Instructions Problems marked with "**Collaboration Allowed**" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.

On problems marked "**No Collaboration**," collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.

An arbitrary subset of these questions will be graded.

Reading You are responsible for reading sections 9.1 to 9.4 in the book

For brevity, write $\sigma_p, \sigma_c, \sigma_r$ for the point (i.e., eigenvalues), continuous, and residual spectrum, respectively, so $\rho(A) = \mathbb{C} \setminus \sigma(A)$ and $\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$.

Problem 1: **Collaboration Allowed** Problem 9.1: let $A \in \mathcal{B}(\mathcal{H})$, then prove $\rho(A^*) = \overline{\rho(A)}$ (where $\overline{\rho(A)}$ is as defined in the book, and does not mean the closure in this case).

Problem 2: **No Collaboration** Problem 9.2: if λ is an eigenvalue of some $A \in \mathcal{B}(\mathcal{H})$, then by the previous problem $\overline{\lambda}$ is in the spectrum of A^*. What can you say about the type of spectrum (of A^*) that $\overline{\lambda}$ belongs to? e.g., for each of ($\sigma_p(A^*), \sigma_c(A^*)$ or $\sigma_r(A^*)$), either give an example to show that it is possible $\overline{\lambda}$ is in the type of spectrum, or prove it is impossible.

Problem 3: **No Collaboration** Problem 9.3: let $A \subset \mathcal{B}(\mathcal{H})$ and $\lambda, \mu \in \rho(A)$. Prove $R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$.

Problem 4: **No Collaboration** Problem 9.4: if P is an orthogonal projection, prove the spectrum is either $\{0\}$ (iff $P = 0$), or $\{1\}$ (iff $P = I$), or else $\{0, 1\}$.

Problem 5: **No Collaboration** Problem 9.5: if $A = A^* \in \mathcal{B}(\mathcal{H})$ is self-adjoint and non-negative on a complex Hilbert space $\mathcal{H}$, prove $\sigma(A) \subset [0, \infty)$. (Don't just prove $\sigma_p(A) \subset [0, \infty)$).

Problem 6: **Collaboration Allowed** Problem 9.8 (modified): spectrum of the right-shift operator S on $\ell^2(\mathbb{Z})$. Show

a) $\sigma_p(S) = \emptyset$
b) $\lambda I - S$ has full range if $|\lambda| < 1$
c) $\lambda I - S$ has full range if $|\lambda| > 1$
d) Therefore the spectrum must be contained in the unit circle $|\lambda| = 1$. Show, in fact, the spectrum is all of the unit circle, and is purely continuous (not residual).
e) By symmetry, conclude the spectrum of the left-shift operator on $\ell^2(\mathbb{Z})$ is identical.
f) Is S a normal operator (on $\ell^2(\mathbb{Z})$)?

Hint: it may be useful consider the Fourier domain for some parts of the question.

Problem 7: **Collaboration Allowed** Problem 9.10: spectrum of the left- and right-shift operators T and S on $\ell^2(\mathbb{N})$ (see book). *Hint: you may wish to solve parts (a)–(g) out-of-order.*

Notation: let $B = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$ be the unit disc (so $\partial B = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ is the unit circle, and $\text{int}B = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$ and $B^c = \mathbb{C} \setminus B$).

Specifically, show:

a) $\rho(S) = B^c$, i.e., $\sigma(S) = B$

b) $\sigma_c(S) = \partial B$

c) $\sigma_r(S) = \text{int}B$

d) $\rho(T) = B^c$, i.e., $\sigma(T) = B$

e) $\sigma_c(T) = \partial B$

f) $\sigma_p(T) = \text{int}B$

g) $\sigma_r(T) = \emptyset$

Problem 8: Collaboration Allowed Problem 9.7 about the Volterra operator. Let $K : L^2(I) \to L^2(I)$, $I = [0, 1]$, be the integral operator defined by

$$(Kf)(x) = \int_0^x f(y)\, dy.$$

a) Find the adjoint operator K^*

b) Show that $\|K\| = 2/\pi$.

c) Show the spectral radius of K is 0.

d) Show $0 \in \sigma_c(K)$.

Hints:

For part (b), use $\sqrt{\cos(\frac{\pi}{2}x)}$. Talk to the instructor for more hints on part (b).
For part (c), use Prop. 9.7 and partial integration.
For part (d), one way to show that 0 cannot be an eigenvalue is by rewriting the integral equation as an ODE (similar problems have occurred on the analysis prelims). Note that the Volterra operator *is* compact, so you could use the spectral theorem for normal operators, but you'd need to prove that it is compact first (hint: Arzela-Ascoli).