Homework 4
APPM 5450 Spring 2018 Applied Analysis 2

Due date: Monday, Feb 19 2018, before 1 PM **Instructor**: Prof. Becker
Theme: Weak convergence in Hilbert Space, and unitary operators

Instructions Problems marked with "**Collaboration Allowed**" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.

On problems marked "**No Collaboration**," collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.

An arbitrary subset of these questions will be graded.

Reading You are responsible for reading section 8.6 of the book, especially Thm. 8.40

Facts Some of the following facts may be useful (you can use these without proof). These are from Terry Tao's 245b class (note 11, exercise 27). Let $(A_n) \subset \mathcal{B}(\mathcal{H})$ be a sequence of bounded linear operators and (x_n) and (y_n) be sequences in $\mathcal{H}$, then $A_n \to 0 \ldots$

a) ... in the *operator norm* sense iff $\langle A_n x_n, y_n \rangle \to 0$ for any *bounded* sequences (x_n), (y_n).

b) ... in the *strong* (operator) sense iff $\langle A_n x_n, y_n \rangle \to 0$ for any *convergent* sequence (x_n) and *bounded* sequence (y_n).

c) ... in the *weak* (operator) sense iff $\langle A_n x_n, y_n \rangle \to 0$ for any *convergent* sequences (x_n), (y_n).

Note: here, $x_n \in \mathcal{H}$ is a vector on it's own. For example, in the special case of $\mathcal{H} = \ell^2$, we could write $x_n = (x_n^{(1)}, x_n^{(2)}, \ldots) \in \ell^2$.

Make sure you can reconcile the following facts: (1) we say $x_n \rightharpoonup 0$ if for all $y \in \mathcal{H}$, $\lim_{n\to\infty} \langle y, x_n \rangle = 0$ (but perhaps $\|x_n\|$ does not converge to zero), yet (2) if $\langle y, x \rangle = 0$ for all $y \in \mathcal{H}$ then $x = 0$.

Problem 1: **Collaboration Allowed** Let T be the left-shift operator on $\ell^2(\mathbb{N})$, i.e., $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. Define the sequence $(T_n)_{n\in\mathbb{N}}$ by $T_n = T^n$, so T_n is the "shift-n-to-the-left" operator. Answer the following and provide justification for your answers:

a) Does $T_n \to 0$ uniformly?

b) Does $T_n \to 0$ in the strong operator sense?

c) Does $T_n \to 0$ in the weak operator sense?

d) Does $T_n^* \to 0$ uniformly?

e) Does $T_n^* \to 0$ in the strong operator sense?

f) Does $T_n^* \to 0$ in the weak operator sense?

g) Which of your answers above would change if we considered $\ell^2(\mathbb{Z})$ instead of $\ell^2(\mathbb{N})$? (No justification needed)

Problem 2: **No Collaboration** Let $\mathcal{H} = \ell^2(\mathbb{N})$ and e_n be the canonical basis vectors. Which of the following sequences converge weakly? Which have strongly convergent subsequences, and which have weakly convergent subsequences?

a) $x_n = n\, e_n$.

b) $y_n = n^{-1/2} \sum_{j=1}^{n} e_j$.

c) $z_n = e_n + e_m$ where $m = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$.

Problem 3: **No Collaboration** If $U : \mathcal{H}_1 \to \mathcal{H}_2$ is unitary, prove $\|U\| = 1$.

Problem 4: **No Collaboration** Let A denote a self-adjoint operator on a Hilbert space $\mathcal{H}$. Let u be any element in $\mathcal{H}$ and set $u_n = e^{inA}u$. Prove that (u_n) has a weakly convergent subsequence. *Hint: you probably want to use the* ***Eberlein-Šmulian theorem****, which says that in a Hilbert space, a set is weakly compact if and only if it is weakly sequentially. Recall that compactness and sequential compactness are the same thing in a metric space (Thm. 1.62 in our book), but the weak topology is not metrizable, so this is not automatically true for weak compactness.*

Problem 5: **Collaboration Allowed** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, and $U : \mathcal{H}_1 \to H_2$ a unitary operator, and $A_1 \in \mathcal{B}(\mathcal{H}_1)$ a self-adjoint operator. Define $A_2 \in \mathcal{B}(\mathcal{H}_2)$ by $A_2 = UA_1U^{-1}$. Prove that A_2 is self-adjoint.

Problem 6: **Collaboration Allowed** Consider the Hilbert space $\mathcal{H} = L^2([-\pi, \pi])$ and the sequence of functions $\varphi_n(x) = x^2 \sin(n\,x)$. Does (φ_n) converge strongly in $\mathcal{H}$? Does it converge weakly? If you answer yes to either question, what is the limit?

Note: to save you some work, you may use the fact (proven in Example 5.47 in the book, or a special case of the Riemann-Lebesgue lemma): if p is a polynomial, and $[a, b]$ a bounded interval in $\mathbb{R}$, then

$$\lim_{n\to\infty} \int_a^b p(x) \sin(nx)\, dx = 0$$

(and the same result holds for cos instead of sin). Proof is via repeated integration-by-parts.