Homework 3
APPM 5450 Spring 2018 Applied Analysis 2

Due date: Friday, Feb 9 2018, before 1 PM **Instructor**: Prof. Becker
Theme: Projections and Bounded Linear Operators on Hilbert Space

Instructions Problems marked with "**Collaboration Allowed**" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.

On problems marked "**No Collaboration**," collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.

An arbitrary subset of these questions will be graded.

Reading You are responsible for reading sections 8.2 – 8.4 of the book.

Problem 1: **No Collaboration** Prove that if P is a projection on a Hilbert space $\mathcal{H}$, then the following three statements are equivalent:

a) P is orthogonal, i.e., $\ker(P) = \operatorname{ran}(P)^\perp$. [Do not use the book's definition that P is orthogonal iff P is a self-adjoint projection]

b) P is self-adjoint

c) $\|P\| = 0$ or 1

Hint: if you wish to prove $c \implies a$, you may want to consider $x \in \operatorname{ran}(P)$ and $y \in \ker(P)$, and examine $\|x - ty\|^2$ as $t \to 0$ and t positive or negative depending on $\langle x, y\rangle$.

Problem 2: **No Collaboration** Problem 8.4: (P_n) is a sequence of orthogonal projections on $\mathcal{H}$ that have nested ranges and $\cup_{n=1}^{\infty} \operatorname{ran}(P_n) = \mathcal{H}$. Prove P_n converges strongly to the identity, but does not converge in the operator norm unless $P_n = I$ for all n sufficiently large.

Problem 3: **No Collaboration** 8.6 from the book, second part only (show an invertible linear map from Hilbert space $\mathcal{H}_1$ to Hilbert space $\mathcal{H}_2$ is unitary iff its inverse is unitary).

Problem 4: **No Collaboration** 8.15 parts a, b and e only.

Definition The book provides a definition of a **coercive functional** $f : \mathcal{H} \to \mathbb{R}$. More generally, if $A \in \mathcal{B}(\mathcal{H})$, we say it is a **coercive operator** if there is a constant $c > 0$ such that $\langle Ax, x\rangle \ge c\|x\|^2$ for all $x \in \mathcal{H}$. We typically require $A = A^*$ before considering coerciveness, since otherwise it's not necessarily true that $\langle Ax, x\rangle$ is even a real number. Note that a coercive operator, which is also assumed to be bounded and self-adjoint, generates a norm $\|x\|_A = \sqrt{\langle x, Ax\rangle}$ that is therefore equivalent to the norm of the Hilbert space, since

$$c\|x\|^2 \le \|x\|_A^2 = \langle x, Ax\rangle \le \|x\|\|Ax\| \le \|x\|^2\|A\|.$$

Problem 5: **No Collaboration** Let $\mathcal{H}$ be a complex Hilbert space and $(\varphi_n)_{n\in\mathbb{N}}$ an orthonormal basis for $\mathcal{H}$. Given a bounded sequence of complex numbers $(\lambda_n)_{n\in\mathbb{N}}$, define the operator A by setting

$$Au = \sum_{n=1}^{\infty} \lambda_n \langle \varphi_n, u\rangle \varphi_n.$$

a) Prove $\|A\| = \sup_n |\lambda_n|$ (and don't just prove $\|A\| \le \sup_n |\lambda_n|$)

b) Prove $A^*u = \sum_{n=1}^{\infty} \overline{\lambda_n} \langle \varphi_n, u \rangle \varphi_n$. Concluded that A is self-adjoint if and only if all λ_n are real.

c) When is A skew-symmetric? When is it non-negative/positive/coercive?

Problem 6: **Collaboration Allowed** Consider the Hilbert space $\mathcal{H} = L^2([-\pi, \pi])$ and the operator $A \in \mathcal{B}(\mathcal{H})$ defined by $(Au)(x) = |x|u(x)$.

a) Prove A is self-adjoint and positive, but not coercive.

b) Prove that

$$\langle u, v \rangle_A \stackrel{\text{def}}{=} \langle Au, v \rangle$$

is an inner product on H

c) Prove that the norm induced by this inner product generates a topology that is *not* equivalent to the topology generated by the L^2-norm.

Fact 1 **Farkas' lemma**. This is a classic result from the early 20th century that underlies the duality theory of linear programming. The proof uses a separating hyperplane result that is trivial in finite dimensions, and in infinite dimensional spaces, it follows from the Hahn-Banach theorem. Farkas' lemma (aka Farkas' Alternative) is stated in finite dimensions. It says that for any matrix $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$, exactly one of the following conditions hold:

a) The set $\{x \mid Ax = b,\ x \ge 0\}$ is non-empty

b) The set $\{y \mid y^T A \ge 0,\ y^T b < 0\}$ is nonempty

Problem 7: **Collaboration Allowed**. **Existence of an equilibrium distribution in finite-state Markov chains** Let $P \in \mathbb{R}^{n \times n}$ be the transition matrix of a Markov chain. It is a stochastic matrix, which means that

$$(\forall\, i, j = 1, \ldots, n)\ (P)_{ij} \ge 0, \quad \text{and} \quad P^T \mathbb{1} = \mathbb{1}$$

i.e., each entry is non-negative and the columns sum to one. The vector of all ones is written $\mathbb{1}$. The Markov chain describes the evolution of a probability vector x_t, for $t = 1, 2, \ldots$, where each $x_t \in \mathbb{R}^n$ is non-negative and sums to 1 (i.e., $\mathbb{1}^T x_t = 1$). The probability vector evolves according to $x_{t+1} = Px_t$. A basic question is when does x_t converge? If it does converge, it must clearly converge to a fixed point $x = Px$ (this x is also known as the equilibrium distribution). Thus an even more basic question is when is there a fixed point solution? *Prove, using the Farkas' alternative, that there exists an equilibrium solution*, e.g., there is a x such that

$$Px = x, \quad x \ge 0, \quad \mathbb{1}^T x = 1$$

(In particular, note that $x = 0$ is not a valid distribution since it cannot be normalized to sum to 1). Stronger guarantees can be made using the Perron-Frobenius theorem.

Fact 2 **Fredholm alternative in finite dimensions**. For any matrix $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$, exactly one of the following conditions hold:

a) The set $\{x \mid Ax = b\}$ is non-empty

b) The set $\{y \mid y^T A = 0,\ y^T b \ne 0\}$ is nonempty

What this is saying is that $\mathbb{R}^m = \operatorname{ran}(A) \oplus \ker(A^*)$ (no need to worry about closure of $\operatorname{ran}(A)$ in finite dimensions, since it is always closed), and therefore if $b \in \operatorname{ran}(A)$ and hence $b \perp \ker(A^*)$, so we have condition (a), or else not, so b is not perpendicular to everything in $\ker(A^*)$ (condition (b)).

⋆ **Optional** Deduce the Fredholm alternative from the Farkas alternative. Hint: you may want to consider the matrix (in Matlab notation) $[A, -A]$.

Proof of Farkas' lemma (in case you were curious)

If both alternatives occur, then $x \geq 0$ and $y^T A \geq 0$, so $y^T Ax = y^T b \geq 0$, but this violates $y^T b < 0$. Thus at most one alternative occurs. We only need to show that if alternative (a) does not hold, the alternative (b) does.

So supposing (a) is false, then the point b does not lie in the cone $C \subset \mathbb{R}^m$ generated by the columns of A. (The columns of A are a set of vectors, and taking all non-negative combinations of these vectors, i.e., Ax for $x \geq 0$, generates a cone). It is a fact that this cone is closed and convex. If $b \notin C$, then there is a plane that strictly separates any closed convex cone C from $b \notin C$ (in infinite dimensions, this would follow from the Hahn-Banach theorem), e.g., there is some $y \in \mathbb{R}^m$ and $\beta \in \mathbb{R}$ such that

$$\begin{aligned} & y^T z + \beta > 0 \quad \forall z \in C \\ & y^T b + \beta < 0. \end{aligned}$$

Pick any $x \geq 0$, then for any $\lambda > 0$, define $z = A(\lambda x)$ so $z \in C$ and hence $\lambda y^T Ax + \beta > 0$. Divide this by λ and take the limit as $\lambda \to \infty$ to conclude

$$y^T Ax \geq 0$$

which was true for all $x \geq 0$, in particular we can pick x to be any of the unit vectors, and thus

$$y^T A \geq 0.$$

Since $0 \in C$, then $y^T 0 + \beta > 0$ so $\beta > 0$. Thus

$$y^T b < -\beta < 0$$

and thus we have proved that y satisfies the second alternative. *Proof from Joel Franklin*

You can also prove Farkas' lemma using duality theory for linear programming (or prove duality theory for linear programming using Farkas' lemma!)