Homework 2
APPM 5450 Spring 2018 Applied Analysis 2

Due date: Friday, Feb 2 2018, before 1 PM **Instructor**: Prof. Becker
Theme: Fourier Series and Projections

Instructions Instructions: Problems marked with "**Collaboration Allowed**" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.
On problems marked "**No Collaboration**," collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.
An arbitrary subset of these questions will be graded.

Reading You are responsible for reading sections 7.2 and 8.1 of the book, and the handout on the website about convergence of Fourier series.

Problem 1: Collaboration Allowed

a) Use Fourier series of the function $f(x) = x$ to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$. You do not have to re-calculate the Fourier coefficients of f since you did this on last week's homework. Can you give two quick proofs that $f \notin H^1(\mathbb{T})$?

b) Use the Fourier series of the function $g(x) = |x|$ to show $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. See the instructor if you need a hint. Convince yourself (in several ways if possible) that $g \in H^1(\mathbb{T}) \setminus H^2(\mathbb{T})$.

c) Use the Fourier series of the function $h(x) = \operatorname{sign}(x)$ to evaluate $S = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$. [This was on the January 2015 prelim exam]

d) On $I = [-\pi, \pi]$, $f(x) = x$ is very smooth, i.e., $f \in C^\infty(I)$, whereas $g(x) = |x|$ is not a smooth, i.e., $g \in C(I) \setminus C^1(I)$. Do you expect the Fourier coefficients of f to decrease faster than those of g? Is this true?

Problem 2: Collaboration Allowed

a) For $f \in L^p(\mathbb{T})$ and $h \in \mathbb{R}$, let $f_h(x) = f(x+h)$ denote the translation of f by h. If $1 \le p < \infty$, show that $f_h \to f$ in $L^p(\mathbb{T})$ as $h \to 0$. Hint: approximate f by a continuous function. [Note: an analogous result does not make sense on $L^p([0,1])$ since our translation would move f outside of $[0,1]$. However, we can consider it on $L^p(\mathbb{R})$ and the result is still true; use, e.g., Theorem 12.50)].

b) Give a counter-example to show this is not true in $L^\infty(\mathbb{T})$.

$\star$ (No work required, just something to ponder) Make sure you can reconcile the fact that the Fourier series of a general $L^2(\mathbb{T})$ (or even $C(\mathbb{T})$) function *do not* converge uniformly, yet the set of trigonometric polynomials is dense in $L^2(\mathbb{T})$ with respect to the uniform norm.

$\star$ (No work required, just something to ponder) Let $a_{m,n} = (1/n)^{1/m}$.
Is $\lim_{n\to\infty}(\lim_{m\to\infty} a_{m,n}) = \lim_{m\to\infty}(\lim_{n\to\infty} a_{m,n})$?

Problem 3: No Collaboration Problem 8.1 parts (a) and (b). If M is a linear subspace of a linear space X, then the *quotient space* X/M is the set $\{x+M \mid x \in X\}$ of affine spaces $x+M \stackrel{\text{def}}{=} \{x+y \mid y \in M\}$ parallel to M. (Note: other common notation for the equivalence class $x+M$ is $[x]$).

a) Show X/M is a vector space, with $(x+M)+(y+M)=(x+y)+M$ and $\lambda(x+M)=(\lambda x)+M$. [To show it is a vector space, you do not need to prove the scalar multiplcation laws (distributive, identity, etc.) nor the commutativity of addition, since these are a bit easy yet tedious.]

b) Let $X=M\oplus N$ and show N is linearly isomorphic to X/M. [The definition of linearly isomorphic is Def. 5.20.]

⋆ (Optional – if you do this, you may collaborate) If the dimension of X/M is finite (the *co-dimension* of M), is every subspace of M necessarily closed? (Assume X is Banach)

⋆ (Optional) If X has a norm, can you think of a norm on X/M? If M is dense in X, what is X/M?

Problem 4: **No Collaboration** Problem 8.2: if $\mathcal{H}=M\oplus N$ is an orthogonal direct sum, show $M^{\perp}=N$ and $N^{\perp}=M$. [Note: by definition, $M\perp N$, so make sure you understand what the problem is asking]

Problem 5: **Collaboration Allowed [Optional]** (This problem is just for fun and will not be graded, so only attempt if you have enough time). The complete solution is given in the first half of section 7.5, but try to solve it without looking at the solutions first.

The problem is to prove that if γ is a closed C^1 curve in the plane ($\mathbb{R}^2$), and of length 2π, then the area enclosed by γ is less than or equal to π, with equality occuring iff γ is a circle. This is a very classic problem, and known as the isoperimetric inequality. In general for d dimensions, given a fixed perimeter or area of the boundary, the maximal volume shape is a sphere, something we all know intuitively.

We parameterize γ using curve-length s as the parameter. Let f and g be functions in $H^1(\mathbb{T})$ such that $\gamma(s)=\big(f(s),g(s)\big)$ so $\gamma:\mathbb{T}\to\mathbb{R}^2$. Recall from Green's theorem that the area A enclosed by the curve is given by

$$A=\frac{1}{2}\int_\gamma (x\,dy-y\,dx)=\frac{1}{2}\int_0^{2\pi}\big(f(s)\dot{g}(s)-g(s)\dot{f}(s)\big)\,ds \tag{1}$$

The problem is to find f and g that maximize A, subject to the constraint that the length of the curve is 2π:

$$\int_0^{2\pi}\big(\dot{f}(s)^2+\dot{g}(s)^2\big)\,ds=2\pi \tag{2}$$

Combining (1) and (2), we have

$$2\pi-A=\int_0^{2\pi}\Big(\dot{f}(s)^2+\dot{g}(s)^2-f(s)\dot{g}(s)+g(s)\dot{f}(s)\Big)\,ds \tag{3}$$

We write f and g as Fourier series with coefficients α_n and β_n ($n\in\mathbb{Z}$), respectively. Now,

a) Use Parseval's relation to rewrite (3) involving the Fourier coefficients

b) Complete the squares to prove that $2\pi-2A$ is non-negative. One good way of completing the squares will involve four squares, two of which are $|n\alpha_n-i\beta_n|^2$ and $|n\beta_n-i\alpha_n|^2$.

c) Finally, prove that equality occurs iff γ is a circle.