Homework 1
APPM 5450 Spring 2018 Applied Analysis 2

Due date: Friday, Jan 26 2018, before 1 PM **Instructor**: Prof. Becker
Theme: Fourier basis on the torus (§7.1), basic Sobolev space on the torus

Instructions Instructions: Problems marked with "**Collaboration Allowed**" mean that collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks. Please write down the names of the students that you worked with.
On problems marked "**No Collaboration**," collaboration with anyone is forbidden. Internet usage is forbidden, but using the course text is allowed, as well as any book mentioned on the syllabus. These problems can be viewed as take-home exams.
An arbitrary subset of these questions will be graded.

Reading You are responsible for reading sections 7.1 and 7.2 of the book.

Problem 1: **No Collaboration** Problem 7.1 from the book: let $\varphi_n(x) = c_n(1+\cos(x))^n$ where c_n is chosen so $\int_{\mathbb{T}} \varphi_n = 1$.

a) Prove that for any $\delta > 0$, $\lim_{n\to\infty} \int_{\delta \le |x| \le \pi} \varphi_n(x)\, dx = 0$.

b) Prove that if the set $\mathbb{P}$ of trigonometric polynomials is dense in $(C(\mathbb{T}), \|\cdot\|_\infty)$, then it is dense in $(C(\mathbb{T}), \|\cdot\|_{L^2})$.

c) Is $\mathbb{P}$ dense in $(C([0, 2\pi]), \|\cdot\|_\infty)$?

Problem 2: **Collaboration Allowed** Problem 7.2 from the book. Suppose $f : \mathbb{T} \to \mathbb{C}$ is a continuous function with Fourier coefficients $\widehat{f}_n$, and let S_N be the partial sum of Fourier terms for $|n| \le N$,

$$S_N = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \widehat{f}_n e^{inx}$$

[Note: in your solutions, you may use Mathematica or similar for manipulations, though you can set the problems up simply enough to avoid requiring Mathematica]

a) Show $S_N = D_N * f$ where D_N is the Dirichet kernel

$$D_N(x) = \frac{1}{2\pi} \frac{\sin\left((N+1/2)x\right)}{\sin(x/2)}$$

(Note: the Dirichet kernel is a periodic version of the sinc kernel; the sinc function arises in the setting of the Fourier *transform* when one truncates the Fourier transform to a finite bandwidth $[-B, B] \subset \mathbb{R}$, analogous to our discrete case of truncating to $(-N, -N+1, \ldots, N-1, N) \subset \mathbb{Z}$.)

b) Let T_N be the mean of the first $N+1$ partial sums, i.e., $T_N = \frac{1}{N+1}(S_0 + S_1 + \ldots S_N)$. Show that $T_N = F_N * f$ where F_N is the Fejer kernel

$$F_N(x) = \frac{1}{2\pi(N+1)} \left(\frac{\sin(N+1)x/2}{\sin(x/2)} \right)^2.$$

c) Which of the families (D_N) and (F_N) are approximate identities as $N \to \infty$? What can you say about the uniform convergence of S_N and T_N to f? For both questions, you do not need to do calculations or a proof, just give some thought to your response.

Problem 3: No Collaboration Problem 7.3 from the book. Let $J = [0, \pi]$.

a) Prove that $\{e_n\}_{n\geq 1}$ is an orthonormal basis for $L^2(J)$ where $e_n = \sqrt{\frac{2}{\pi}}\sin(nx)$. [Note: you do not need to prove that it is orthonormal, just show that it is total]

b) Prove that $\{f_n\}_{n\geq 0}$ is also an orthonormal basis for $L^2(J)$, where $f_0(x) = \sqrt{1/\pi}$ and for $n \geq 1$, $f_n = \sqrt{\frac{2}{\pi}}\cos(nx)$ [Note: you do not need to prove that it is orthonormal, just show that it is total. You do not need to repeat any calculations or arguments that you already made for part (a)]

Problem 4: Collaboration Allowed Problem 7.4 from the book: let $T, S \in L^2(\mathbb{T})$ where $T(X) = |x|$ and $S(x) = 1$ when $0 < x < \pi$ and $S(x) = -1$ when $-\pi < x < 0$.

a) Compute the Fourier series of T and S

b) Show $T \in H^1(\mathbb{T})$ and $T' = S$

c) Show $S \notin H^1(\mathbb{T})$ (and hence $T \notin H^2(\mathbb{T})$)

Problem 5: Optional: Problem 7.6. Suppose $f \in H^1([a,b])$ and $f(a) = f(b) = 0$. Prove the Poincaré inequality:

$$\int_a^b |f(x)|^2\,dx \leq \frac{(b-a)^2}{\pi^2}\int_a^b |f'(x)|\,dx.$$