Final Exam Selected Solutions
APPM 5450 Spring 2015 Applied Analysis 2

Date: Thursday, May. 7 2015
Instructor: Dr. Becker   Your name: ________________

If the mathematical field is not specified, you may assume it is $\mathbb{R}$ or $\mathbb{C}$ at your convenience. The symbol $\mathcal{H}$ denotes an arbitrary Hilbert space. Your proofs may use any major result discussed in class (if you are unsure, please ask).

Total points possible: 104.

N.B. Unlike the homeworks, the grades may be curved. Points are not distributed according to difficulty. When justifying your answer, do as little work as possible, e.g., avoid explicitly evaluating integrals if not necessary. Partial credit is possible.

For problems 1 and 2, PLEASE WRITE DIRECTLY ON THIS SHEET

Problem 1: (20 pts) Definitions. State the following definitions and/or theorems. You may skip one definition (please clearly mark which one should not be graded). 2 points each.

(1) Define the Sobolev space $H^s(\mathbb{T})$ for $s > 0$.

(2) Sobolev embedding theorem, any version *Solution*: $H^k(\mathbb{R}) \subset C(\mathbb{R})$ for $k > 1/2$ (careful, you should really specify $\mathbb{R}$ since for $\mathbb{R}^d$ you need $k > d/2$). Don't confuse this with Riemann-Lebesgue! It's not in $C_0(\mathbb{R})$.

(3) Banach-Alouglu theorem, any variant

(4) Briefly define the Fourier transform on $\mathcal{S}(\mathbb{R})$ (the space of Schwartz functions). *Solution*: Via the standard integral definition.

(5) Briefly define the Fourier transform on $\mathcal{S}(\mathbb{R})^*$. *Solution*: In a weak sense, e.g., for $T \in \mathcal{S}(\mathbb{R})^*$ define $\widehat{T}$ s.t. for all $\varphi \in \mathcal{S}$, $\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle$.

(6) Briefly define the Fourier transform on $L^2(\mathbb{R})$. *Solution*: The standard answer is that it is defined by using the density of $\mathcal{S}$ and extending this via the BLT theorem. Valid alternatives: it is a diagonal operator with respect to the Hermite functions, mapping coefficients $(c_n)_{n=0,1,\ldots}$ to $((-i)^n c_n)_{n=0,1,\ldots}$. If you define it via the integral form, -1 points (the integral form is valid only if $f \in L^1$ or $f \in \mathcal{S}$). Note that your answer should never contain any sums, only integrals. Remember: for $L^2(\mathbb{R})$, use the Fourier transform and integrals, while for $L^2(\mathbb{T})$, use Fourier series and sums.

(7) What does it mean for $(\varphi_n) \subset \mathcal{S}$ to converge to a limit φ? *Solution*: It means for all α, β multi-indices, then $\|\varphi_n - \varphi\|_{\alpha,\beta} \to 0$. The pseudo-norm is defined in eq. (11.3).

(8) Lebesgue dominated convergence theorem

(9) Fubini's theorem

(10) Define what it means for a function $f : X \to Y$ to be measurable with respect to measure spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$.

(11) In the above definition, if $Y = \overline{\mathbb{R}}$ and $\mathcal{B}$ is the Borel σ-algebra generated by the standard topology on $\overline{\mathbb{R}}$, what is a simplified definition of a measurable function? *Solution*: That for all $c \in \mathbb{R}$, the set $\{x \mid f(x) < c\}$ is measurable. (Equivalently, sets like $\{x \mid f(x) \geq c\}$, etc., are also valid).

Problem 2: (30 pts) Mark true/false (or yes/no). No justification needed. $\mathcal{H}$ denotes a Hilbert space. 2 points each.

Solution: Most students did very well on this, but almost no one got question 6 correct.

(1) Let $C, D \subset \mathcal{H}$. If $C = D^\perp$, is $C^\perp = D$? *Solution*: False. This is only true if D is closed. For example, let $\mathcal{H} = R^2$ and $D = \{qe_1 \mid q \in \mathbb{Q}\}$ where e_1 is the first canonical unit vector. Then $C = \text{span}(e_2)$ so $C^\perp = \text{span}(e_1) = \overline{D} \neq D$.

(2) If $\mu(X) < \infty$, then $L^p(X) \subset L^q(X)$ if $p \le q$. *Solution*: False.

(3) If $\mu(X) < \infty$, then $L^p(X) \subset L^q(X)$ if $p \ge q$. *Solution*: True.

(4) If $\mu(X) = \infty$, then $L^p(X) \subset L^q(X)$ if $p \le q$. *Solution*: False.

(5) If $\mu(X) = \infty$, then $L^p(X) \subset L^q(X)$ if $p \ge q$. *Solution*: False.

(6) Is the Heaviside function $H(x) = \chi_{(0,\infty)}(x)$ weakly differentiable? *Solution*: No, since if it were, it would violate the Sobolev embedding theorem, since weakly differentiable implies it is in H^1 [sorry for overloading the "H" symbol] but it is clearly not continuous. See example 11.13 in the book for more discussion. It has a distributional derivative but not a weak derivative, and this is why we say that sometimes ∂T_f is not the same as $T_{\partial f}$.

(7) Is it possible that in $\ell^1(\mathbb{N})$, weak convergence always implies strong convergence? *Solution*: Yes, and in fact it is true (Schur 1924). If $1 < p < \infty$, this is not possible since ℓ^p is reflexive, so the Banach-Alouglu theorem says the unit ball is weakly compact, and the unit ball in an infinite dimensional space is never strongly compact. For $p = 1$, the space is not reflexive so this is not precluded (this is what the question is getting at).

Note that we discussed this on problem 4b in HW 14 from last semester.

(8) $L^p(\mathbb{R})$ is separable for all $1 \le p \le \infty$. *Solution*: False, since this is not true for $p = \infty$.

(9) $L^p([0,1])$ is separable for all $1 \le p \le \infty$. *Solution*: False, since this is not true for $p = \infty$.

(10) $C([0,1])$ is dense in $L^\infty([0,1])$. *Solution*: False, since that would imply it is separable.

(11) $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$. *Solution*: True.

(12) $C_c(\mathbb{R})$ is complete with respect to the uniform norm. *Solution*: False, for example a Cauchy sequence may converge to e^{-x^2}.

(13) Let $f \in L^2(\mathbb{T})$ and define the partial sum $f_N = \sum_{n=-N}^{N} \widehat{f}_n e_n$ where (e_n) is the Fourier basis. Does $\|f - f_n\|_{L^2} \to 0$? *Solution*: Yes.

(14) If $f, g \in L^1(\mathbb{R})$, is $fg \in L^1(\mathbb{R})$? *Solution*: No.

(15) If $f, g \in L^1(\mathbb{R})$, is $f * g \in L^1(\mathbb{R})$? *Solution*: Yes, follows from Fubini's theorem.

Problem 3: (12 pts) Short response. 2 points each. For examples of functions, don't forget to specify their domain.

(1) The space $L^2(\mathbb{T})$ contains periodic functions, but the functions are not defined pointwise because they are really equivalence classes. If they are not defined pointwise, how can they be periodic? Briefly discuss.

Solution: Because continuous functions are defined pointwise, we can easily define $C(\mathbb{T})$. Then define $L^2(\mathbb{T})$ as the closure of $C(\mathbb{T})$ under the L^2 norm.

(2) On $I = [-\pi, \pi]$, $f(x) = x$ is very smooth, i.e., $f \in C^\infty(I)$, whereas $g(x) = |x|$ is not a smooth, i.e., $g \in C(I) \setminus C^1(I)$. Do you expect the Fourier coefficients of f to decrease faster than those of g? Is this true?

Solution: If we consider the periodic extension, then f is no longer even continuous, so $f \notin C(\mathbb{T})$, whereas $g \in C(\mathbb{T})$, so we actually expect the Fourier coefficients of g to decrease faster, which is what we observe.

(3) Give an example of an operator that is positive but not coercive. *Solution*: If your example is of a linear operator, then the space is necessarily infinite dimensional. Take $\ell^2(\mathbb{N})$ for example, and define $T(x) = (1/nx_n)$ for $x = (x_n) \in \ell^2$. More generally, use any diagonal operator (in terms of an orthonormal basis, or just on ℓ^2 coordinate-wise) such that $A(x) = (a_n x_n)$ and $a_n > 0$ but $a_n \to 0$.

Another example is example 9.5, using $(Tf)(x) = x \cdot f(x)$ on $L^2([0,1])$. Since $x \in [0,1]$, we see $\langle f, Tf \rangle \ge 0$ always, but this can be arbitrarily small as the support of f clusters around 0.

This question was missed by a lot of students. One of the issues is positive vs. non-negative. We say an operator T is "positive" or "positive definite" if $\langle x, Tx \rangle > 0$ for all x, and it is "non-negative" or "positive semi-definite" if we just have $\langle x, Tx \rangle \ge 0$.

(4) Give an example of functions (f_n) that converge pointwise a.e. to f but $\lim_{n\to\infty} \int f_n \neq \int f$. *Solution*: Many possibilities, e.g., in L^1, take boxes supported on $[0, 1/n]$ with height n so $f = 0$ and $\lim_{n\to\infty} \int f_n = 1 \neq 0 = \int 0$. This question was answered correctly by most students.

(5) Give an example of a function that is Lebesgue integrable but not Riemann integrable.

Solution: The indicator function of the rational numbers.

Note that if a function is very nice and non-negative (so no cancellations happen), then the integrals will be the same. So it's not the case that, say, $f(x) = x^{-1/2}$ over $[0, \infty)$ is Lebesgue integral but not Riemann integral (it's neither). Rather, you need something "funny" happening, such as the indicator of the rationals. For the opposite direction (see next problem) you need cancellation.

(6) Give an example of a function that has an improper Riemann integral but is not Lebesgue integrable. *Solution*: The sinc function on $\mathbb{R}$ on $\mathbb{R}$. The Lebesgue integral does not allow cancellations the same way an improper Riemann integral does. Note that if $\sin(x)/x$ is not integrable on $[0, \infty)$, then neither is $\sin(1/x)/x$ on $[0, \infty)$ since they are the same after changing variables $x \leftarrow 1/x$. The former does not decay fast enough at ∞ while the latter oscillates too much near 0.

You can make lots of examples by exploiting cancellation and using conditionally convergent series. For example, on $[0, \infty)$, let $i(x)$ be the function that returns the greatest integer less than x, e.g., $i(3.7) = 3$. Then the function $f(x) = 1/x \cdot (-1)^{i(x)}$ is modulated and has cancellations, so it has an improper Riemann integral but not a Lebesgue one; the same conclusion holds for the function $f(x) = \sum_{i=n}^{\infty} \frac{1}{n}(-1)^n \chi_{[n,n+1]}$. In particular, for both functions, we exploit the fact that we know from undergrad analysis that the sum $\sum_{n=1}^{\infty} \frac{1}{n}(-1)^n$ is conditionally convergent.

Also note that a lot of the "integrals" for Fourier coefficients that we write are not well-defined in the Lebesgue sense, but we write them that way because it is short-hand notation for the Fourier transform (which we define using *density* arguments if $f \in L^2 \setminus L^1$). Most of these integrals are well-defined in the improper Riemann sense, so these would make examples. For example, $f(x) = 1/x$ on all of $\mathbb{R}$, because the contributions from $x > 0$ and $x < 0$ cancel each-other out.

This question was missed by a lot of students.

Problem 4: (6 pts) Convergence. 3 points each.

(1) Let $f_n(x) = e^{in\pi x} \in L^2([0, 1])$. Does f_n converge strongly, or weakly, and if so, what is the limit? Justify your answer.

Solution: It converges weakly to 0 because the dual space is also $L^2([0, 1])$ and this is a subset of $L^1([0, 1])$ so the Riemann-Lebesgue lemma applies, and the decay property of the Fourier transform implies the weak convergence. It does not converge strongly because $|f_n(x)|^2 = 1$ so $\|f_n\| = 1$. [Note that via Banach-Alouglu, we can immediately infer that it has a weakly convergent subsequence, and you could try to use this fact to prove that the whole sequence must converge.] Or, not the equivalence with Fourier series and use Bessel's inequality. Or note that it is orthonormal and (via Bessel's inequality) we know all orthonormal sequences cannot converge strongly but do converge weakly.

Quite a few students did not correctly calculate $\|f_n\|$, which should be 1. Recall that $|e^{ix}| = 1$ for all x.

Rubric: 2 points for right answer without correct justification, 0 to 1 point for wrong answers depending on arguments.

(2) Let $f_n(x) = e^{in\pi x}\chi_{[-n,n]} \in L^2(\mathbb{R})$. Does f_n converge strongly, or weakly, and if so, what is the limit? Justify your answer.

Solution: The Riemann-Lebesgue lemma no longer applies since the dual space is $L^2(\mathbb{R})$ and this is not a subset of $L^1(\mathbb{R})$. In fact, we do not have weak convergence because a necessary condition is that $\|f_n\|$ is bounded, and it is not (since $\|f_n\|^2 = 2n$). Of course (f_n) does not converge strongly either.

Problem 5: (6 pts) Convergence and Integrals. Let (f_n) and f be integrable functions on $[1, \infty)$ such that $f_n \to f$ a.e. Give a short proof or a counter-example for the following statements: (3 points each)

(1) If $f_n \to f$ uniformly, then $\lim_{n\to\infty} \int_1^\infty f_n = \int_1^\infty f$.

Solution: No. Take $f_n = 2^{-n}\chi_{[2^n,2^{n+1}]}$ so $\int f_n = 1$ and it converges to 0 uniformly, but $\lim_{n\to\infty} \int f_n = 1 \neq 0 = \int 0$. Or take f_n to be 0 outside $[1, n+1]$ and linear inside $[1, n+1]$ with $f(1) = 2/n$ and $f(n+1) = 0$, so that $\int_1^\infty f_n(x)\,dx = 1$ but yet it converges uniformly to 0.

This does not violate the uniform convergence theorem since that theorem requires a bounded domain of integration.

(2) If (f_n) is monotone decreasing, then $\liminf_{n\to\infty} \int_1^\infty f_n = \int_1^\infty f$.

Solution: Yes. Use Fatou's lemma (the "liminf" should be a big hint to use Fatou).

Specifically, to apply Fatou we need non-negative functions. Since $f_n \searrow f$ a.e., we have that $f_n - f$ is non-negative a.e.. We can work directly with f_n instead of $f_n - f$ since we can just subtract off $\int f$ since it is integrable so this is finite. Thus

$$\int f = \int \lim_{n\to\infty} f_n = \int \liminf_n f_n \leq \liminf_n \int f_n$$

We also have $f_1 - f_n$ is non-negative a.e. for all n, and similarly can subtract off $\int f_1$, so another application of Fatou gives

$$-\int f = \int \lim_{n\to\infty} -f_n = \int -\liminf_n f_n \leq \liminf_n \int -f_n$$

and taking the negative out gives $\int f \geq \liminf_n \int f_n$. Combining the two inequalities gives the equality.

We can actually quickly prove a stronger statement: since $-f_n \nearrow -f$ and $-f_n \geq -f_1 \in L^1$, we can apply the MCT, and conclude that in fact $\int f = \lim_{n\to\infty} \int f_n$.

These two problems were taken from part of a question from an old prelim test, and student scores on both these problems were low (only 3 students gave correct or nearly-correct answers to the 2nd question).

Problem 6: (6pts) Spectral theory. Let $A \in \mathcal{B}(\mathcal{H})$. 3 points each.

(1) Prove $\lambda \in \sigma_r(A)$ implies $\overline{\lambda} \in \sigma_p(A^*)$.

Solution: Let $T = A - \lambda I$ so $T^* = A^* - \overline{\lambda} I$. Then

$$\mathcal{H} = \overline{\operatorname{ran}(T)} \oplus \ker(T^*)$$

so $\lambda \in \sigma_r(A)$ implies $\mathcal{H} \neq \overline{\operatorname{ran}(T)}$ and hence there is a non-trivial element in $\ker(T^*)$ which implies $\lambda \in \sigma_p(A^*)$.

This question is fundamental. See Prop. 9.12 in the book.

(2) If $A = A^*$, prove $\sigma_r(A) = \emptyset$.

Solution: If A is self-adjoint, the spectrum is real. By the previous question, we have $\lambda \in \sigma_r(A)$ implies $\lambda = \overline{\lambda} \in \sigma_p(A^*) = \sigma_p(A)$, but $\sigma_r(A) \cap \sigma_p(A) = \emptyset$ by definition, hence we cannot have $\lambda \in \sigma_r(A)$.

Problem 7: (8 pts) Fourier transform. 2 points each.

(1) Let $f(x) = e^{i\omega x}$ for $\omega \in \mathbb{R}$ be a function on $\mathbb{R}$. Which of the following spaces does f live in: $\mathcal{S}(\mathbb{R}), \mathcal{S}^*(\mathbb{R}), L^1(\mathbb{R}), L^2(\mathbb{R})$ (combinations allowed, e.g., "none" or "all")?

Solution: It is in $\mathcal{S}^*(\mathbb{R})$ and none of the others. If we chose $I = [0,1]$ instead of $I = \mathbb{R}$, for example, then it would be in all of $\mathcal{S}(I), \mathcal{S}^*(I), L^1(I), L^2(I)$.

(2) Let $\mathcal{F}$ be the Fourier transform. What is $\widehat{f} \stackrel{\text{def}}{=} \mathcal{F}(f)$ for f as above? *Solution*:

An acceptable answer is δ_ω (i.e. $\delta(\cdot - \omega)$), though $\sqrt{2\pi}\delta_\omega$ is more precise.

Note that the review sheet mentioned two key Fourier transform pairs: the sinc function and box function, and the Gaussian function (and a scaled version of itself). This δ_ω and $e^{i\omega x}$ is another key pair you should memorize forever.

For intuition, use eq (11.33) which states that $\delta(x) = 1/\sqrt{2n} \int e^{ikx}\, dk$, though this is not meant in a precise sense since this is not Lebesgue integrable. Working with the integral themselves is messy since you need to use an ultraviolet cutoff like Prop. 11.29. It is better to use Prop. 11.27 which says $\widehat{e^{i\omega x}\varphi} = \tau_\omega \widehat{\varphi}$ for $\varphi \in \mathcal{S}$, and therefore this also holds for $\mathcal{S}^*$ since it is defined weakly so all properties of the Fourier transform on $\mathcal{S}$ carry over to $\mathcal{S}^*$. Then just use the fact that we know $\widehat{\delta} = 1/\sqrt{2n}$ (example 11.31) and then shift this.

For reference, to use eq (11.33) informally, we do (for $\varphi \in \mathcal{S}$ arbitrary)

$$\begin{aligned}
\langle \widehat{f}, \varphi \rangle = \langle f, \widehat{\varphi} \rangle &= \int f(x)\widehat{\varphi}(x)\, dx \\
&= \int f(x) \left(\int e^{-ixy} \varphi(y)\, dy \right) dx \\
&= \int \varphi(y) \left(\int e^{-ixy} f(x)\, dx \right) dy \\
&= \int \varphi(y) \left(\int e^{ix(\omega - y)}\, dx \right) dy \\
&= \int \varphi(y) \sqrt{2n}\delta(\omega - y)\, dy \\
&= \sqrt{2n} \langle \delta(\omega - \cdot), \varphi \rangle \\
&= \sqrt{2n} \langle \delta(\cdot - \omega), \varphi \rangle.
\end{aligned}$$

(Why is this informal? Because we have changed the order of integration even though Fubini's theorem doesn't apply, and we are using eq (11.33) which is not literally true.)

(3) Let $\varphi \in \mathcal{S}(\mathbb{R})$. What is $\langle \delta', \varphi \rangle$?

Solution: It is $-\varphi'(0)$. See Example 11.14 in the book.

(4) Let $\varphi \in \mathcal{S}(\mathbb{R})$. What is $\delta' * \varphi$?

Solution: First, observe that we are looking for the answer to be a function, not a scalar. The equation above Example 11.19 in the book defines the convolution as $\delta' * \varphi = \langle \delta', R\tau_x \varphi \rangle$ where R is the reversal operation and τ_x the shift-by-τ, so if we have $\varphi(y)$ then $(\tau_x \varphi)(y) = \varphi(y - x)$ and $(R\tau_x \varphi)(y) = \varphi(x - y)$. The action of the distribution is $\langle \delta', \psi \rangle = -\langle \delta, \psi' \rangle = -\psi'(0)$ (see previous question). If $\psi = R\tau_x \varphi$ then via the chain rule, $\psi'(y) = -\varphi'(x - y)$ thus $(\delta' * \varphi)(x) = -\varphi'(x)$, so the answer is $-\varphi'$.

This example is very important. When you convolve a function with the delta function, you get back the function—see references on Green's function.

Most students missed this problem. Convolutions were not emphasized in this class but they are fair game on the prelim, as they are mentioned in our text and they are considered part of undergrad analysis.

Problem 8: (3 pts) Given an example of a measurable space (X, μ) and measurable sets (E_n) such that $E_1 \supseteq E_2 \supseteq E_3 \ldots$ and

$$\lim_{n \to \infty} \mu(E_n) \neq \mu\left(\bigcap_n E_n \right).$$

Solution: The necessary ingredient in the counter-example is that some E_n has infinite measure. For example, let $X = \mathbb{R}$ and $E_n = [n, \infty)$ so $\bigcap E_n = \emptyset$.

Th infinite measure ingredient is "necessary" since on problem 2 from homework 11, we proved that if any E_n has finite measure, then we do have equality.

Problem 9: (13 pts) Bounded linear operators.

(1)* (1 pt) Let X be a Banach space. If $\varphi(x) = \varphi(y)$ for all $\varphi \in X^*$, prove $x = y$.

Solution: This is problem 5.6 from the book and was problem 1 in HW 10 from last semester. You first define a functional ψ on the subspace spanned by x (wlog let $y = 0$ since φ is linear) such that $\psi(x) = \|x\|$ (unless $x = 0$), then use Hahn-Banach to extend this to a functional on the whole space. Therefore if $\varphi(x) = 0$ for all functions φ, we have $\psi(x) = 0$ but also $\psi(x) = \|x\|$ if $x \neq 0$, so this is a contradiction.

(2) (2 pts) A *closed* operator $T : X \to Y$, X and Y normed linear spaces, is such that if for $(x_n) \subset X$, $x_n \to x$ and $T(x_n) \to y$, then $T(x) = y$. Explain how this differs from a *continuous* operator, and state whether closed operators are continuous, or vice-versa, or neither.

Solution: This is weaker than being continuous since we pre-suppose that $T(x_n)$ converges. For continuous (i.e., sequentially continuous), the fact that $T(x_n)$ converges (if x_n converges) is a consequence, not a condition. Hence continuous implies closed, but not vice-versa.

(3) (0 pts, fact) Let $\mathcal{H}$ be a Hilbert space, and let $A : \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{H} \to \mathcal{H}$ be operators (not necessarily linear nor bounded) with the property that $\langle Ax, y\rangle = \langle x, By\rangle$ for all $x, y \in \mathcal{H}$. Then you can prove A (and hence B) must be linear operators.

(4) (3 pts) Under the same assumptions as part (3), prove A (and hence B) must be bounded as well. Hint: you may use the following corollary of the open-mapping theorem known as the "Closed Graph Theorem": if X and Y are Banach and $A : X \to Y$ is linear, then A is closed iff it is bounded.

Solution: We wish to show A is closed, so we can apply the theorem and conclude it is also bounded. Let $x_n \to x$ and $A(x_n) \to y$, and we seek to prove $y = A(x)$. Then for all $x \in \mathcal{H}$,

$$\begin{aligned} \langle y, z\rangle = \lim_{n\to\infty} \langle Ax_n, z\rangle &= \lim_{n\to\infty} \langle x_n, Bz\rangle \\ &= \langle x, Bz\rangle \\ &= \langle Ax, z\rangle \end{aligned}$$

since the inner product is continuous. Using part (a), we conclude $y = Ax$ as required.

(5) (4 pts) Let $H^1(\mathbb{T})$ be the Sobolev space on the torus. 2 points each:

i. Define the weak derivative (denote the operator by D).
ii. Is $D : H^1(\mathbb{T}) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$ bounded? Briefly justify

Solution: We can define it weakly such that if $f \in H^1$, then for all $\varphi \in C^1(\mathbb{T})$, $\langle T', \varphi\rangle = -\langle T, \varphi'\rangle$. Or, define it on the Fourier coefficients and map $\widehat{f_n}$ to $(in)\widehat{f_n}$.

It is not bounded because as we see via the Fourier coefficient definition, it increases the magnitude of coefficients.

(6) (3 pts) In chapter 10, which we did not cover, the book defines a concept called the "formal adjoint" of the weak derivative D. Can this be the same concept of "adjoint" that we discussed this semester? Please discuss why or why not.

Solution: We just argued in 5(ii) that D is not bounded, but we proved in 4 that any operator with an adjoint mus be bounded. Therefore D cannot have an adjoint in the sense we defined. The "formal adjoint" is different since it changes the domain. Basically, a linear operator in the sense we have discussed in class must be defined on the whole domain, but you could restrict it. This is related to how we view H^1: is can be a subset of L^2 (that is, using the L^2 norm), which is not complete, or it can be its own Hilbert space with the H^1 norm (in which case it is complete).