APPM 4360/5360 Introduction to Complex Variables and Applications

HOMEWORK #5

Assigned: Monday March 4, 2019

DUE: At class Monday March 18, 2019

XC: Extra Credit

1. Consider $f_n(z)g_n(z)$ in a domain D where: $|g_n(z)| \leq M, M$ constant in D and $f_n(z)$ converges to zero uniformly for all $z \in D$. Prove $f_n(z)g_n(z)$ converges to zero uniformly for all $z \in D$.

2. Find the Taylor series expansion of the following functions:
a) $\frac{z^2}{1+z^3}$, $|z| < 1$, b) $\cosh kz$, $k > 0$ constant, $c) ze^{ikz^2}$, $k > 0$ constant.

3. Given:

$$F(z) = \int_{-\infty}^{\infty} f(t)e^{izt}dt$$

where $f(t) = e^{\alpha_1 t}, t < 0, f(t) = e^{-\alpha_2 t}, t > 0, \alpha_j > 0, j = 1, 2$ constants. Find the region of the complex plane where $F(z)$ is analytic; explain. Do the same if $f(t) = te^{-\kappa t^2}, \kappa > 0$; explain. $F(z)$ is referred to as the Fourier transform of $f(t)$.

4. Let

$$F(z) = \int_{0}^{\infty} f(t)e^{-zt}dt$$

where $f(t)$ is continuous and $|f(t)| \leq Ae^{-\alpha t}, A > 0, \alpha > 0$, constants. Find the region of the complex plane where $F(z)$ is analytic; explain. $F(z)$ is referred to as the Laplace transform of $f(t)$.

5. Let $f(z) = \frac{1}{z^2+\alpha^2}, \alpha > 0$. Find the Laurent expansion in the regions

a) $|z| > \alpha$ b) $|z| < \alpha$

6. Given the function

$$f(z) = \frac{2z}{(z-i)(z+2)}$$

Find the Laurent series of $f(z)$ in the regions: a) $|z| < 1$, b) $1 < |z| < 2$, c) $|z| > 2$

7. Discuss all singularities of the following functions; including the type of singularity: pole–include order, essential, branch point, cluster ..., that each of these functions have in the finite $z-$plane. For parts a,b,c,d if the functions have a Laurent series around any of the singularities find the first two nonzero terms

a) $\sec z$ b) $\frac{1}{e^z-1}$ c) $\frac{\log z}{z(z-2)}$ d) $\sin(1/z^2)$ e) $\coth(1/z)$

8. Solve: Evaluate the integral $\frac{1}{2\pi i}\oint_C f(z)dz$ where C is the unit circle centered at the origin and $f(z)$ is given below

$a)$ $\frac{z^2}{z^2+a^2}$, $0 < a < 1$, $b)$ $\cot(2z)$ $c)$ $\frac{\log(z+a)}{z+1/a}$, $a > 1$, principal branch

9. a. Let $f(z) = 1 + z^2 + z^4 + ..., |z| < 1$. Find a function, call it $g(z)$, that analytically continues $f(z)$ to $|z| > 1$; what can be said about $g(z)$ on $|z| = 1$ and for $|z| < 1$; explain.

b. Consider $f(z) = \log(2(z-1))$, $z-1 = re^{i\theta}$. Discuss/explain the analytic continuation of the function from $R_1 \to R_2 \to R_3$ where $r > 0$ and θ is in the regions: $R_1 : 0 \le \theta \le \pi/2$; $R_2 : \pi/3 \le \theta \le 4\pi/3$; $R_3 : \pi \le \theta \le 7\pi/3$.

XC 3.3.6