1. Consider \( f_n(z)g_n(z) \) in a domain \( D \) where: \(|g_n(z)| \leq M, M \) constant in \( D \) and \( f_n(z) \) converges to zero uniformly for all \( z \in D \). Prove \( f_n(z)g_n(z) \) converges to zero uniformly for all \( z \in D \).

2. Find the Taylor series expansion of the following functions:
   a) \( z^2 \frac{1}{1+z^3}, |z| < 1 \), b) \( \cosh kz, k > 0 \) constant, c) \( ze^{ikz^2}, k > 0 \) constant.

3. Given:
   \[ F(z) = \int_{-\infty}^{\infty} f(t)e^{izt}dt \]
   where \( f(t) = e^{\alpha_1 t}, t < 0, f(t) = e^{-\alpha_2 t}, t > 0, \alpha_j > 0, j = 1, 2 \) constants. Find the region of the complex plane where \( F(z) \) is analytic; explain. Do the same if \( f(t) = te^{-\kappa t^2}, \kappa > 0 \); explain. \( F(z) \) is referred to as the Fourier transform of \( f(t) \).

4. Let
   \[ F(z) = \int_{0}^{\infty} f(t)e^{-zt}dt \]
   where \( f(t) \) is continuous and \(|f(t)| \leq Ae^{-\alpha t}, A > 0, \alpha > 0 \), constants. Find the region of the complex plane where \( F(z) \) is analytic; explain. \( F(z) \) is referred to as the Laplace transform of \( f(t) \).

5. Let \( f(z) = \frac{1}{z^2 + \alpha^2}, \alpha > 0 \). Find the Laurent expansion in the regions
   a) \(|z| > \alpha\), b) \(|z| < \alpha\)

6. Given the function
   \( f(z) = \frac{2z}{(z-i)(z+2)} \)
   Find the Laurent series of \( f(z) \) in the regions: a) \(|z| < 1\), b) \(1 < |z| < 2\), c) \(|z| > 2\)
7. Discuss all singularities of the following functions; including the type of singularity: pole–include order, essential, branch point, cluster ..., that each of these functions have in the finite \( z \)-plane. For parts a,b,c,d if the functions have a Laurent series around any of the singularities find the first two nonzero terms

a) \( \sec z \)

b) \( \frac{1}{e^{z-1}} \)

c) \( \frac{\log z}{z(z-2)} \)

d) \( \sin(1/z^2) \)

e) \( \coth(1/z) \)

8. Solve: Evaluate the integral

\[
\frac{1}{2\pi i} \oint_C f(z) \, dz
\]

where \( C \) is the unit circle centered at the origin and \( f(z) \) is given below

a) \( \frac{z^2}{z^2+a^2}, \quad 0 < a < 1 \)

b) \( \cot(2z) \)

c) \( \frac{\log(z+a)}{z+1/a}, \quad a > 1, \quad \text{principal branch} \)

9. a. Let \( f(z) = 1 + z^2 + z^4 + ..., \quad |z| < 1 \). Find a function, call it \( g(z) \), that analytically continues \( f(z) \) to \( |z| > 1 \); what can be said about \( g(z) \) on \( |z| = 1 \) and for \( |z| < 1 \); explain.

b. Consider \( f(z) = \log(2(z-1)), \quad z-1 = re^{i\theta} \). Discuss/explain the analytic continuation of the function from \( R_1 \rightarrow R_2 \rightarrow R_3 \) where \( r > 0 \) and \( \theta \) is in the regions: \( R_1: 0 \leq \theta \leq \pi/2; \quad R_2: \pi/3 \leq \theta \leq 4\pi/3; \quad R_3: \pi \leq \theta \leq 7\pi/3. \)

XC 3.3.6