APPM 4360/5360: Introduction to Complex Variables
Spring 2019
Homework 1

Problem 1: (15 points) Express each of the following complex numbers in polar exponential form: $re^{i\theta}$

(a) -2i

$$-2i = 2e^{3i\pi 2} \quad \Longrightarrow \quad r = 2,\ \theta = 3\pi/2$$

(b) $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$$r^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 = 1, \qquad \tan\theta = \frac{-1/\sqrt{2}}{1/\sqrt{2}} = -1.$$

Since $\cos\theta = x > 0$, $\theta = 7\pi/4$. Thus, $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = e^{7\pi i/4}$.

(c) $\sqrt{3} - i$

$$r^2 = (\sqrt{3})^2 + (-1)^2 = 4, \qquad \tan\theta = -\frac{1}{\sqrt{3}}.$$

Since $x > 0, y < 0$, then $\theta = 2\pi - \pi/6 = 11\pi/6$. Thus, $\sqrt{3} - i = 2e^{11\pi i/6}$.

Problem 2: (15 points) Express the following in the form $x + yi$, where x and y are real:

(a) $\frac{1}{1-2i}$

$$\frac{1}{1-2i} = \frac{(1+2i)}{(1-2i)(1+2i)} = \frac{1+2i}{5} \quad \Longrightarrow \quad x = 1/5,\ y = 2/5$$

Thus, $\frac{1}{1-2i} = 1/5 + (2/5)i$.

(b) $(1-i)^2(1+2i)$

$$(1-i)^2(1+2i) = (1-2i-1)(1+2i) = -2i(1+2i) = 4-2i.$$

(c) $|1-3i|$

This is a real number, the absolute value of $1-3i$, i.e.

$$|1-3i| = \sqrt{1^2+(-3)^2} = \sqrt{10}.$$

Problem 3: (12 points) Solve for all the roots of the following equation: $z^3 - 2z^2 + 2z = 0$

Solution: First note that $z^3 - 2z^2 + 2z = z(z^2 - 2z + 2) = 0$, so one root is $z_1 = 0$. Now solve for the other two

$$z_{2,3} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

Thus, the 3 roots are $z = 0, 1+i, 1-i$.

Problem 4: (16 points) Establish the following inequalities:

(a) $|4z_1 - z_2| \leq 4(|z_1| + |z_2|)$

By triangle inequality,

$$|4z_1 - z_2| \leq |4z_1| + |-z_2| = 4|z_1| + |z_2| \leq 4(|z_1| + |z_2|).$$

(b) $|2z_1\bar{z}_2 + 3\bar{z}_1 z_2| \leq 5|z_1||z_2|$

By triangle inequality,

$$|2z_1\bar{z}_2 + 3\bar{z}_1 z_2| \leq 2|z_1\bar{z}_2| + 3|\bar{z}_1 z_2| = 2|z_1||z_2| + 3|z_1||z_2| = 5|z_1||z_2|.$$

Problem 5: (15 points) Sketch the region associated with the following inequality and determine if the region is open, closed, bounded, compact, connected: $6 \leq |3z+7| \leq 9$; Explain.

Solution: Rewrite the inequalities as $2 \le |z + 7/3| \le 3$, which shows that this is an annular region between 2 circles with center $z = -7/3$ and radii 2 and 3, including both the outer boundary circle $|z + 7/3| = 3$ and the inner boundary circle $|z + 7/3| = 2$. The region is not open since it contains its boundary; it is closed; it is bounded (can be surrounded by a finite circle); it is compact since closed and bounded; it is connected since its every two points can be connected by a curve completely lying inside the set.

Problem 6: (10 points) Show that $\Im(1/z)$ and $\Im(-z)$ have the same sign for all $z \neq 0$.
Solution: Let $z = x + iy$, i.e. $\Re z = x$ and $\Im z = y$. Then

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}.$$

Thus, $\Im(1/z) = -y/(x^2 + y^2)$ which has the same sign as $-y = \Im(-z)$.

Problem 7: (12 points) Find the series expansion around $z = 0$ of: $\frac{\sin z - z}{z^2}$
Solution:

$$\sin z = \sum_{n\ge 0} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z + \sum_{n\ge 1} \frac{(-1)^n z^{2n+1}}{(2n+1)!},$$

therefore

$$\frac{\sin z - z}{z^2} = \sum_{n\ge 1} \frac{(-1)^n z^{2n-1}}{(2n+1)!} = \sum_{n\ge 0} \frac{(-1)^{n+1} z^{2n+1}}{(2n+3)!}.$$

Problem 8: (40 points) Evaluate the following limits, explain reasoning:

(a) $\lim_{z\to 0} \frac{\cos(\beta z)-1}{z^2}$, $\beta \neq 0$ constant

Using series expansion of cos,

$$\lim_{z\to 0} \frac{\cos(\beta z) - 1}{z^2} = \lim_{z\to 0} \frac{1 - (\beta z)^2/2 + \cdots - 1}{z^2} = \lim_{z\to 0}\left(-\frac{\beta^2}{2} + \ldots\right) = -\beta^2/2.$$

(Dots stand for powers of z higher than those written out.)

(b) $\lim_{z\to 0} \frac{\sin(\alpha z)}{\sin(\beta z)}$, α, $\beta \neq 0$ constant

$$\lim_{z\to 0} \frac{\sin(\alpha z)}{\sin(\beta z)} = \lim_{z\to 0} \frac{\alpha z - (\alpha z)^3/6 + \dots}{\beta z - (\beta z)^3/6 + \dots} = \lim_{z\to 0} \frac{\alpha z(1 - (\alpha z)^2/6 + \dots)}{\beta z(1 - (\beta z)^2/6 + \dots)} = \frac{\alpha}{\beta}.$$

(c) $\lim_{z\to\infty} \frac{Mz^4+z}{(Nz^2+3)^2}$; $M, N \neq 0$

$$\lim_{z\to\infty} \frac{Mz^4 + z}{(Nz^2+3)^2} = \lim_{z\to\infty} \frac{z^4(M + 1/z^3)}{z^4(N + 3/z^2)^2} = \lim_{z\to\infty} \frac{M + 1/z^3}{(N + 3/z^2)^2} = \frac{M}{N}.$$

(d) $\lim_{z\to\infty} \frac{\sinh 2az}{\cosh 2az}$, $a > 0$ constant

$$\frac{\sinh 2az}{\cosh 2az} = \frac{e^{2az} - e^{-2az}}{e^{2az} + e^{-2az}},$$

consider two different ways of approaching $z = \infty$: first let $z = x$ real and $x \to +\infty$, then

$$\lim_{x\to+\infty} \frac{e^{2ax} - e^{-2ax}}{e^{2ax} + e^{-2ax})} = \lim_{x\to+\infty} \frac{e^{2ax}}{e^{2ax}} = 1.$$

On the other hand, if still $z = x$ but now $x \to -\infty$, then

$$\lim_{x\to-\infty} \frac{e^{2ax} - e^{-2ax}}{e^{2ax} + e^{-2ax})} = \lim_{x\to-\infty} \frac{-e^{-2ax}}{e^{-2ax}} = -1.$$

The two limits are different which shows that $\lim_{z\to\infty} \frac{\sinh 2az}{\cosh 2az}$ does not exist.

Problems 9: (40 points)

(a) **Problem 1.3.3:** (20 points) If $|g(z)| \leq M$, $M > 0$ for all z in a neighborhood of $z = z_0$, show that if $\lim_{z\to z_0} f(z) = 0$, then

$$\lim_{z\to z_0} f(z)g(z) = 0.$$

Solution: Consider $\lim_{z\to z_0} |f(z)g(z)| = \lim_{z\to z_0} |f(z)||g(z)|$. We have

$$0 \le \lim_{z\to z_0} |f(z)||g(z)| \le \lim_{z\to z_0} M|f(z)| = M \lim_{z\to z_0} |f(z)| = 0,$$

i.e. $0 \le \lim_{z\to z_0} |f(z)g(z)| \le 0$ which implies that $\lim_{z\to z_0} |f(z)g(z)| = 0$ and then $\lim_{z\to z_0} f(z)g(z) = 0$.

(b) (20 points) Where are the following functions differentiable: i) $\tanh z$; ii) $e^{1/(z-i)}$

i) **Solution:**

$$\tanh z = \frac{\sinh z}{\cosh z},$$

a ratio of two functions each of which is differentiable everywhere in $\mathbb{C}$. Therefore $\tanh z$ is also differentiable everywhere except for the points where

$$\cosh z = 0 \quad \Longrightarrow \quad e^z + e^{-z} = 0 \quad \Longrightarrow \quad e^{2z} = -1 = e^{i\pi + 2\pi i n},\ n \in \mathbb{Z},$$

i.e. except for points $z = i(\pi/2 + n\pi)$, $n \in \mathbb{Z}$.

ii) **Solution:**

$e^{1/(z-i)} = e^{g(z)}$, where $g(z) = 1/(z-i)$, i.e. $e^{1/(z-i)}$ is a composition of two functions $f(g(z))$. Here $f(g) = e^g$ is entire (differentiable for all z) and $g(z) = 1/(z-i)$ is differentiable for all z except for $z = i$. Thus, $e^{1/(z-i)}$ is also differentiable for all $z \neq i$.

Problem 10: (25 points) Find the general solution of the following differential equation:

$$\frac{d^3 w}{dz^3} - 8w = 0;$$

write the solution in **real** form.

Solution: Look for solutions in the form $w = e^{kz}$, k constant. Substitute this into the equation and get

$$k^3 - 8 = 0 \qquad \Longrightarrow \qquad k = 2; 2e^{2i\pi/3}; e^{-2i\pi/3},$$

the three solutions for k. Thus, the general solution of the DE is

$$w(z) = c_1 e^{2z} + c_2 \exp(2e^{2i\pi/3} z) + c_3 \exp(2e^{-2i\pi/3} z),$$

where c_1, c_2, c_3 are arbitrary (complex) constants. To get the **real** form we assume now that z is real and express

$$e^{\pm 2i\pi/3} = \cos(2\pi/3) \pm i \sin(2\pi/3) = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2},$$

to rewrite the solution as

$$w(z) = c_1 e^{2z} + e^{-z}(c_2 e^{i\sqrt{3}z} + c_3 e^{-i\sqrt{3}z}) =$$

$$= c_1 e^{2z} + e^{-z} \left(c_2(\cos(\sqrt{3}z) + i\sin(\sqrt{3}z)) + c_3(\cos(\sqrt{3}z) - i\sin(\sqrt{3}z)) \right) =$$

$$= c_1 e^{2z} + e^{-z} \left((c_2 + c_3)\cos(\sqrt{3}z) + i(c_2 - c_3)\sin(\sqrt{3}z) \right).$$

Since the functions of z in the last line are real for real z, to get real solution, one has to take coefficients $A = c_1$, $B = c_2 + c_3$ and $C = i(c_2 - c_3)$ to be real. Thus,

$$w(z) = Ae^{2z} + Be^{-z}\cos(\sqrt{3}z) + Ce^{-z}\sin(\sqrt{3}z)$$

is the real form of the solution.

Extra credit: (10 points)
Use 'ϵ, δ' formulation to prove that $\lim_{z\to i} z^2 = -1$

Solution: We are to prove that for any $\epsilon > 0$ there is $\delta > 0$ such that, for all $|z - i| < \delta$, we have $|z^2 + 1| < \epsilon$.
For $|z - i| < \delta$, use the inequalities

$$|z^2 + 1| = |z - i||z + i| = |z - i||z - i + 2i| \leq |z - i|(|z - i| + 2) < \delta(2 + \delta)$$

We see that, for a given $\epsilon > 0$, it is enough to have $\delta(2 + \delta) < \epsilon$, e.g. for $0 < \epsilon < 1$, it is enough to take e.g. $\delta = \epsilon/3$, then $\delta(2 + \delta) < (\epsilon/3) \cdot 3 = \epsilon$, so also $|z^2 + 1| < \epsilon$.