Problem 1: (20 points) Answer each part of this question with TRUE or FALSE. DO NOT write T or F. No justification for your answers is required.

(a) A scalar differential equation of order $n$
$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$$
can be written as a first-order system of differential equations with $n$ dependent variables.
(b) If $\lambda$ is an eigenvalue of an invertible matrix $A$, then $-\lambda$ is an eigenvalue of $A^{-1}$.
(c) The set $\{x, x^2 - 1, 3x^2 + 2\}$ is a basis for $P_2$.
(d) If the equation $A\vec{x} = \vec{b}$ has a solution, then $A$ is invertible.

Problem 2: (20 points)

(a) Show that the Laplace transform of
$$f(t) = \begin{cases} 0, & t < 1 \\ e^{1-t} \sin(t-1), & t \geq 1 \end{cases}$$
is
$$F(s) = \frac{e^{-s}}{(s+1)^2 + 1}$$
(b) Use the Laplace transform and part (a) to determine the unique solution to the following initial value problem:
$$\ddot{x} + 2\dot{x} + 2x = \delta(t - 1), \quad x(0) = 0, \quad \dot{x}(0) = 0.$$
(c) Plot your solution from part (b). Interpret the behavior of the solution in the context of a spring-mass system.

Problem 3: (20 points) Consider the differential equation
$$2y'' + 4y' + 10y = 3e^{-t} \cos(2t) \quad (1)$$

(a) Calculate the general solution to the associated homogeneous equation for the differential equation (1).
(b) Verify that the functions that form a basis to the homogeneous solution space that you calculated in part (a) are linearly independent.
(c) Using Variation of Parameters, calculate a particular solution to the differential equation (1). Credit will not be awarded for solutions to part (c) using the method of Undetermined Coefficients. (Hint: $\cos^2(x) = \frac{1}{2} \cos(2x) + \frac{1}{2}$). Simplify your solution.
(d) Give the general solution to the differential equation (1).

Problem 4: (20 points)

Consider the initial value problem
$$y' + y = \frac{1}{y}, \quad y(0) = y_0 \quad (2)$$

(a) Classify the differential equation (2) with at least 3 characteristics.
(b) For what initial values $y_0$ does Picard’s Theorem guarantee the existence of a unique solution?
(c) Determine whether the differential equation (2) has any equilibrium points and draw a phase line diagram. Denote stable equilibria with a filled circle, unstable equilibria with an open circle, and singularities with an X (locations where the DE is undefined).
(d) Use separation of variables to solve the IVP for $y_0 = 2$.

Problem 5: (20 points) Consider the system of differential equations

$$\vec{x}' = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \vec{x} \quad (3)$$

(a) Compute the eigenvalues and eigenvectors for the matrix in equation (3).
(b) Give a basis for the nullspace of the matrix in equation (3). That is, provide a basis for the set of all vectors that satisfy the equation $A\vec{x} = \vec{0}$, where $A$ is given above.
(c) Give the general solution to the system of differential equations (3).
(d) Compute the equilibrium point for the system of differential equations (3) and give its stability.

Some Laplace transforms and properties:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) \, dt$$

$$\mathcal{L}\{1\} = \frac{1}{s} \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2 + b^2} \quad \mathcal{L}\{\cos(bt)\} = \frac{s}{s^2 + b^2}$$

$$\mathcal{L}\{\text{step}(t-a)\} = \frac{e^{-as}}{s} \quad \mathcal{L}\{f(t-c)\text{step}(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\} \quad \mathcal{L}\{\delta(t-a)\} = e^{-as}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \quad \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \quad \mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}$$