Problem 1: (20 points) Answer each part of this question with TRUE or FALSE. DO NOT write T or F. No justification for your answers is required.

(a) The method of undetermined coefficients is an appropriate technique for calculating a particular solution to the differential equation \( t^2y'' + y = t^3 \).

(b) The differential equation \( a_3y''' + a_2y'' + a_1y' + a_0y = 0 \) could have three linearly independent trigonometric solutions.

(c) The general solution of the differential equation \( y'' + 9y = \sin(9t) \) is bounded for all \( t \in \mathbb{R} \).

(d) Solutions of the differential equation \( \ddot{x} + bx' + x = 9 \cos(t) \) can always be decomposed into a steady-state part and a transient part.

Solution:

(a) False, the method of undetermined coefficients is only appropriate for linear DEs with constant coefficients.

(b) False, trigonometric solutions correspond to complex roots, but a cubic characteristic equation can have at most 2 complex roots since they must occur in conjugate pairs.

(c) True,

\[
y(t) = c_1 \cos(3t) + c_2 \sin(3t) - \frac{1}{72} \sin(9t)
\]

(d) False, solutions are unbounded for \( b \leq 0 \).

Problem 2: (20 points) Consider the differential equation

\[
y'''(t) + 5y'' + 4y' - 10y = f(t)
\]  

(a) Find the general associated homogeneous solution to the differential equation (1).

Hint: \( y(t) = e^t \) is an associated homogeneous solution to the differential equation (1).

(b) For the following forcing functions, provide the correct ‘guess’ for calculating a particular solution via the method of undetermined coefficients. Do Not Solve For The Values Of The Undetermined Coefficients.

(i) \( f(t) = \cos(t) \)

(ii) \( f(t) = t^3 + e^{3t} \)

(iii) \( f(t) = e^t \)

(iv) \( f(t) = t^2 \sin(2t) \)

(c) Using the method of undetermined coefficients, calculate a particular solution for the differential equation (1) when \( f(t) = -10t^2 - 2t + 4 \).

Solution:

(a) The roots of the characteristic equation are \( \lambda = -3 \pm i, 1 \), so the associated homogeneous solution is \( y_h = c_1 e^{-3t} \cos(t) + c_2 e^{-3t} \sin(t) + c_3 e^t \).

(b) 

(i) \( y_p = b_1 \cos(t) + b_2 \sin(t) \)

(ii) \( y_p = b_3 t^3 + b_2 t^2 + b_1 t + b_0 + b_4 e^{3t} \)

(iii) \( y_p = b t e^t \)

(iv) \( y_p = b_1 t^2 \cos(2t) + b_2 t^2 \sin(2t) + b_3 t \cos(2t) + b_4 t \sin(2t) + b_5 \cos(2t) + b_6 \sin(2t) \)
(c) \[ y_p = b_2 t^2 + b_1 t + b_0 \]
\[ y_p = t^2 + t + 1 \]

**Problem 3:** (20 points) Suppose a block of mass \( m = 1 \) is attached to a spring with spring constant \( k = 4 \) and slides on a surface with frictional constant \( b \). Assume the equilibrium position of the block is at \( x = 0 \). Initially, the block is pulled to \( x = 1 \) and released from rest.

(a) Write down initial conditions and the equation governing the motion of the block, including possible external forces \( f(t) \).

(b) Suppose \( f(t) \equiv 0 \).

(i) Determine the unique solution of the IVP for \( b = 4 \), denote it \( x_4(t) \).

(ii) Determine the unique solution of the IVP for \( b = 5 \), denote it \( x_5(t) \).

(iii) Evaluate the following limit to show that the solution \( x_4 \) of the critically damped system (part (i), \( b = 4 \)) reaches the equilibrium “before” the solution \( x_5 \) of the overdamped system (part (ii), \( b = 5 \)):

\[ \lim_{t \to \infty} \frac{x_4(t)}{x_5(t)} \]

(c) Now suppose \( b = 0 \) and \( f(t) = \cos(2t) \). Determine the unique solution of the IVP and describe the long-term behavior.

**Solution:**

(a) \[ \ddot{x} + bx + 4x = f(t), \quad x(0) = 1, \quad x'(0) = 0 \]

(b) \[ r^2 + br + 4 = 0 \quad \Rightarrow \quad r_{1,2} = \frac{1}{2} \left( -b \pm \sqrt{b^2 - 16} \right) \]

(i) \( r = -2 \quad \Rightarrow \quad x_4(t) = (c_1 + c_2 t) e^{-2t} = (1 + 2t) e^{-2t} \)

(ii) \( r_{1,2} = -1, -4 \quad \Rightarrow \quad x_5(t) = c_1 e^{-t} + c_2 e^{-4t} = \frac{4}{3} e^{-t} - \frac{1}{3} e^{-4t} \)

(iii) \[ \lim_{t \to \infty} \frac{x_4(t)}{x_5(t)} = \lim_{t \to \infty} \frac{3(1 + 2t) e^{-2t}}{4 e^{-t} - e^{-4t}} = \lim_{t \to \infty} \frac{\frac{6}{4e^t + 2e^{-2t}}}{\text{L'Hôpital's Rule}} = 0 \]

(c) Pure resonance: the solution becomes unbounded as \( t \to \infty \).

\[ x(t) = \cos(2t) + \frac{1}{4} t \sin(2t) \]

**Problem 4:** (20 points) Consider the differential equation

\[ t^2 y'' - ty' + y = \frac{3}{2} t^{5/2} \]  \hspace{1cm} (2)

for \( t > 0 \). Both \( y_1(t) = t \) and \( y_2(t) = t \ln t \) are solutions of the associated homogeneous equation.

(a) Show that \( y_1 \) and \( y_2 \) are linearly independent for \( t > 0 \).

(b) Use the method of variation of parameters to find a particular solution of the differential equation (2).

(c) Find the unique solution of the differential equation (2) satisfying \( y(1) = 0 \) and \( y'(1) = 1 \).

**Solution:**

(a) The Wronskian is nonzero:

\[ W[y_1, y_2](t) = \left| \begin{array}{cc} t & t \ln t \\ 1 & 1 + \ln t \end{array} \right| = t > 0 \]
(b) Standard form:

\[ y'' - \frac{1}{t} y' + \frac{1}{t^2} y = \frac{3}{2} \sqrt{t} \quad \Rightarrow \quad f(t) = \frac{3}{2} \sqrt{t} \]

\[ v_1(t) = -\int \frac{y_2 f}{W} \, dt = -\frac{3}{2} \int \sqrt{t} \ln t \, dt = -t^{3/2} \ln t + \frac{2}{3} t^{3/2} \]

\[ v_2(t) = \int \frac{y_1 f}{W} \, dt = \frac{3}{2} \int \sqrt{t} \, dt = t^{3/2} \]

\[ y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) = \frac{2}{3} t^{5/2} \]

(c) \[ y(t) = c_1 t + c_2 t \ln t + \frac{2}{3} t^{5/2} = \frac{2}{3} \left( t^{5/2} - t \right) \]

Problem 5: (20 points)

(a) Explain why the formula for \( v_1(t) \) in the method of variation of parameters will not result in division by zero.

(b) Suppose \( \lambda_1 = 3 \) and \( \lambda_2 = 3 \) are two roots of the characteristic equation for a linear, homogeneous differential equation with constant coefficients. Explain why \( y_1 = e^{3t} \) and \( y_2 = e^{3t} \) are not the corresponding functions in the basis of the homogeneous solution space.

Solution:

(a) The equation has division by the Wronskian of the basis functions for the homogeneous solution space, which are linearly independent.

(b) The solutions to the homogeneous equation must be linearly independent. The provided solutions are not linearly independent. The second solution should be \( y_2 = te^{3t} \).