Problem 1: (20 points) Answer each part of this question with TRUE or FALSE. DO NOT write T or F. No justification for your answers is required.

(a) Let \( A \) be a matrix such that \( \det(A) = 5 \). The determinant of \( A \)'s reduced row echelon form must be 1.

(b) If the rank of an \( m \times n \) matrix is \( n \), then the system \( A\vec{x} = \vec{b} \) has exactly one solution.

(c) If \( A \) is an \( n \times n \) invertible matrix, then the rows of \( A \) form a basis for \( \mathbb{R}^n \).

(d) A basis for \( \mathbb{P}_3 \), the set of polynomials of degree 3 or less, is \( \{ t^3 - t^2 - t - 1, t^2 + t + 2, 1 \} \).

Solution:

(a) True
(b) False. Inconsistencies can still occur for the overdetermined case, \( m > n \), e.g.

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\vec{x}
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

(c) True
(d) False. This cannot be a spanning set since \( \dim(\mathbb{P}_3) = 4 \).

Problem 2: (20 points) Consider the system of algebraic equations

\[
\begin{align*}
x_1 + 2x_2 + 3x_3 &= 7 \\
2x_1 + x_2 - 2x_3 &= 3 \\
3x_1 - 7x_3 &= -1
\end{align*}
\]

(a) Write the system of algebraic equations (1) as a single matrix equation, \( A\vec{x} = \vec{b} \), where

\[
\vec{x} = 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\].

Clearly identify the matrix \( A \) and the vector \( \vec{b} \).

(b) Use Gauss-Jordan elimination to solve the matrix equation \( A\vec{x} = \vec{0} \) for all possible solutions, using the matrix \( A \) and vector \( \vec{x} \) from part (a).

(c) The vector \( \vec{x}_p = 
\begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix}
\) is a solution to the matrix equation \( A\vec{x} = \vec{b} \) from part (a). Give all possible solutions to the matrix equation \( A\vec{x} = \vec{b} \) from part (a).

(d) Verify your solution to part (c).

Solution:

(a) \[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 1 & -2 \\
3 & 0 & -7
\end{bmatrix}
\begin{bmatrix}
\vec{x}
\end{bmatrix}
= 
\begin{bmatrix}
7 \\
3 \\
-1
\end{bmatrix}
\]

\( A \) is the matrix, \( \vec{x} \) is the vector
Problem 3: (20 points) Consider the matrix
\[
A = \begin{bmatrix}
1 - \lambda & 3 & 4 \\
3 & 3 - \lambda & 3 \\
2 & 0 & -1 - \lambda
\end{bmatrix}
\]  
(2)

(a) Identify good choices of rows or columns to use while calculating the determinant by the method of cofactors and explain why.

(b) Calculate the determinant of \( A \) via cofactors.

(c) Find all values of \( \lambda \) such that \( A\vec{v} = \vec{0} \) has at least two solutions. Explain your choice of values.

Solution:

(a) The third row or the second column would be good because the zeros would reduce the total work required.

(b) \( \det (A) = \lambda (\lambda + 3) (\lambda - 6) \)

(c) \( \lambda = 0, -3, 6 \) all make \( \det (A) = 0 \), meaning that \( A\vec{v} = \vec{0} \) has infinitely many solutions.

Problem 4: (20 points) Consider the matrix
\[
A = \begin{bmatrix}
1 & 3 & -4 & 1 \\
4 & 2 & -5 & 5 \\
5 & -5 & 2 & 7
\end{bmatrix}
\]

(a) What is the rank of \( A \)?

(b) Find a basis and the dimension of the column space of \( A \).

(c) Find a basis and the dimension of the row space of \( A \),

\[
\text{span} \{ \begin{bmatrix} 1 & 3 & -4 & 1 \\ 4 & 2 & -5 & 5 \\ 5 & -5 & 2 & 7 \end{bmatrix} \}.
\]

(d) The null space of a matrix \( A \) is the set of all vectors \( \vec{x} \) with \( A\vec{x} = \vec{0} \), i.e.

\[
\left\{ \vec{x} \in \mathbb{R}^4 \mid A\vec{x} = \vec{0} \right\}.
\]
Find a basis and the dimension of the null space of \( A \).

**Solution:**

The REF of \( A \) is

\[
\begin{bmatrix}
1 & 3 & -4 & 1 \\
0 & 1 & -\frac{11}{10} & -\frac{1}{10} \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(a) \( \text{rank } A = 2 \)

(b) \( \text{dim (col } A) = 2 \) with basis \( \left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \right\} \)

(c) \( \text{dim (row } A) = 2 \) with basis \( \left\{ \begin{bmatrix} 1 & 3 & -4 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -\frac{11}{10} & -\frac{1}{10} \end{bmatrix} \right\} \) or \( \left\{ \begin{bmatrix} 1 & 3 & -4 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 2 & -5 & 5 \end{bmatrix} \right\} \)

(d) The RREF of \( A \) is

\[
\begin{bmatrix}
1 & 0 & -\frac{7}{10} & \frac{13}{10} \\
0 & 1 & -\frac{11}{10} & -\frac{1}{10} \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

so

\[\vec{x} = s \begin{bmatrix} \frac{7}{10} \\ \frac{1}{10} \\ 1 \end{bmatrix} + t \begin{bmatrix} -\frac{13}{10} \\ -\frac{1}{10} \\ 0 \end{bmatrix}\]

and \( \text{dim (null } A) = 2 \) with basis \( \left\{ \begin{bmatrix} 7 \\ 11 \\ 10 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 10 \end{bmatrix} \right\} \).

**Problem 5:** (20 points) The following parts are unrelated:

(a) Determine whether or not the set of functions \( \left\{ \begin{bmatrix} \sin(2t) \\ \sqrt{2} \sin(t) \end{bmatrix}, \begin{bmatrix} \sqrt{2} \cos(t) \\ \cos(2t) \end{bmatrix} \right\} \) is linearly independent on all of \( \mathbb{R} \).

(b) Consider the differential equation \( y^{(3)} + 6y^{(2)} + 11y' + 6y = 0 \) (3)

(i) Verify the following three solutions of (3):

\[y_1(t) = e^{-t}, \quad y_2(t) = e^{-2t}, \quad y_3(t) = e^{-3t}\]

Suggestion: Define \( y_j(t) = e^{-jt} \) and plug in \( j = 1, 2, 3 \) later.

(ii) Define

\[y(t) = c_1y_1(t) + c_2y_2(t) + c_3y_3(t)\]

for some \( c_1, c_2, c_3 \in \mathbb{R} \). Is \( y \) also a solution of (3)?

(iii) Determine whether or not the set of functions \( \{y_1(t), y_2(t), y_3(t)\} \) is linearly independent on all of \( \mathbb{R} \).

**Solution:**
(a) The functions will be linearly independent if the only \( c_1, c_2 \) satisfying
\[
c_1 \begin{bmatrix} \sin(2t) \\ \sqrt{2}\sin(t) \end{bmatrix} + c_2 \begin{bmatrix} \sqrt{2}\cos(t) \\ \cos(2t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
for all \( t \) are \( c_1 = c_2 = 0 \). For \( t = \pi/4 \) we have
\[
c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
Since \( \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \), this implies \( c_1 = c_2 = 0 \) so the functions are linearly independent.

(b) (i) \( y_j^{(3)} + 6y_j^{(2)} + 11y_j' + 6y_j = e^{-jt} (-j^3 + 6j^2 - 11j + 6) = 0 \)
for \( j = 1, 2, 3 \).

(ii) Yes, the superposition principle applies since the DE is linear homogeneous.

(iii) The set \( \{y_1, y_2, y_3\} \) is linearly independent since the Wronskian is never zero:
\[
W[y_1, y_2, y_3](t) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{-2t} & e^{-3t} \\ -e^{-t} & -2e^{-2t} & -3e^{-3t} \\ e^{-t} & 4e^{-2t} & 9e^{-3t} \end{vmatrix} = -e^{-6t} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = -e^{-6t}((18 - 12) - (9 - 3) + (4 - 2)) = -2e^{-6t} \neq 0
\]