ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor’s name, (3) your recitation section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized 1 side only) crib sheet is allowed.

**Problem 1:** (20 points) Answer each part of this question with TRUE or FALSE. DO NOT write T or F. No justification for your answers is required.

(a) Suppose the solution to a differential equation is approximated numerically using a second order method with a step size of \( h = 0.3 \). When reducing the step size to \( h = 0.1 \) we would expect error to approximately reduce by a factor of 9.

(b) Picard’s theorem states that the differential equation \( y' = t^3/y \) does not have a unique solution near the point \((3, 0)\).

(c) The equilibria of the coupled system of equations

\[
\frac{dx}{dt} = 7x - 3xy \\
\frac{dy}{dt} = -y + 7xy
\]

are \((x, y) = (0, 7/3)\) and \((1/7, 0)\).

(d) If \(q_1(t)\) and \(q_2(t)\) are both solutions of the equation \(\sin(t)q' = \cos(t)q\), then \(q_1(t) + k q_2(t)\) is also a solution for any real constant \(k\).

**Solution:**

(a) True
(b) False
(c) False
(d) True

**Problem 2:** (20 points) Consider the differential equation

\[
\frac{dy}{dt} = \frac{y^2 + t^2}{yt}
\]

(a) Define a new variable \(v = y/t\). Rewrite Eq. (1) as a differential equation for the new variable \(v\) (still with independent variable \(t\)).

(b) Use separation of variables to find the general solution of the rewritten DE for \(v\).

(c) Using \(y = tv\) and your solution from part (b), determine the unique solution satisfying the original DE (1) for \(y\) and \(y(1) = -2\).

(d) Demonstrate that your solution from part (c) indeed satisfies Eq. (1) for \(t \geq 1\).

**Solution:**

(a) \(y' = \frac{y}{t} + \frac{t}{y} = v + \frac{1}{v} \Rightarrow v + tv' = v + \frac{1}{v} \Rightarrow v' = \frac{1}{tv}\)

(b) \(\frac{1}{2}v^2 = \ln|t| + C \Rightarrow v(t) = \pm \sqrt{2 \ln|t| + 2C}\)

(c) \(y(t) = tv(t) = \pm t\sqrt{2 \ln|t| + 2C}\)

\(-2 = y(1) = -\sqrt{2C} \Rightarrow C = 2 \Rightarrow y(t) = -t\sqrt{2 \ln|t| + 4}\)
Problem 3: (20 points) Consider the differential equation
\[
\frac{dy}{dt} = ty + t^3
\]  
(2)
(a) Identify all possible techniques to find the general solution to Eq. (2).
(b) Find the solution to the associated homogeneous equation of Eq. (2) using separation of variables.
(c) Find a particular solution to Eq. (2).
(d) Find the general solution to Eq. (2).
(e) Find the unique solution to Eq. (2) that passes through \((t = \sqrt{2}, y = 5)\).

Solution:

(a) Both Euler-Lagrange with Variation of Parameters and Integrating Factor can be used.
(b) \( y' = ty \)
\[
\int \frac{1}{y} dy = \int t dt
\]
\[
y = c_1 e^{t^2/2}
\]
(c) \( y_p = v(t) e^{t^2/2} \)
\[
v(t) = \int e^{-t^2/2} t^3 dt
\]
\[
\int t^3 e^{-t^2/2} dt = -t^2 e^{-t^2/2} + \int 2te^{-t^2/2}
\]
\[
v(t) = -e^{-t^2/2} (t^2 + 2)
\]
\[
y_p = -t^2 - 2
\]
(d) \( y = c_1 e^{t^2/2} - t^2 - 2 \)
(e) \( y(\sqrt{2}) = 5 = c_1 e^{1} - 2 - 2 \)
\[
c_1 = 9e^{-1}
\]
\[
y(t) = 9e^{t^2/2-1} - t^2 - 2
\]

Problem 4: (20 points) The following two parts are unrelated:

(a) Consider the differential equation
\[
y' = e^y y^4 - 3e^y
\]  
(3)
Explain if a phase line or direction field is most appropriate for visualizing the equilibrium points of Eq. (3).
(b) Consider the differential equation
\[
\frac{dx}{dt} = tx^3 - tx^2 - 2tx
\]  
(4)
(i) Sketch the direction field for Eq. (4).
(ii) Identify and classify all equilibrium points for Eq. (4).

Solution:

(a) A phase line is more appropriate, as the differential equation is autonomous.
Problem 5: (20 points) The following two parts are unrelated:

(a) A large cooler initially contains 20L of a water and powdered lemonade mixture with a concentration of 5g/L. Thinking the lemonade is too sweet, you decide to pour in a lemonade mixture with a concentration 1g/L at a rate of 3L/min, while draining the uniformly mixed contents of the cooler at a rate of 2L/min. Note that the volume will depend on time. Set up but do not solve the IVP describing the amount of powdered lemonade (in grams) in the cooler after \( t \) minutes. Make sure to clearly define all variables used.

(b) Archaeological excavations have revealed an undiscovered ancient city that was buried by a volcanic eruption and preserved. Wooden beams found in the city are measured to have a concentration of \( ^{14}\text{C} \) that is 64\% of the amount of a living tree. Assuming that \( ^{14}\text{C} \) radioactively decays with a half-life of 5600 years (if your answers involve logarithms, you may leave these unevaluated):

(i) What is the decay constant?
(ii) How old is the ancient city?
(iii) What percentage of the concentration of \( ^{14}\text{C} \) in a living tree will remain in the beams 5600 years from today?

Solution:

(a) Suppose \( t \) is time in minutes. Let \( y(t) \) denote the amount of powdered lemonade in the cooler in grams at time \( t \), and \( V(t) \) denote the volume of water and lemonade mixture in the cooler at time \( t \). We are given

\[
C_{in} = 1, \quad F_{in} = 3, \quad C_{out} = \frac{y(t)}{V(t)}, \quad F_{out} = 2,
\]

\[
y(0) = 20(5) = 100, \quad V(t) = 20 + (F_{in} - F_{out})t = 20 + t,
\]

where \( C_{in}, C_{out} \) are the concentrations in and out respectively, and \( F_{in}, F_{out} \) are the flow rates in and out respectively. The IVP is

\[
y' = C_{in}F_{in} - C_{out}F_{out} = 3 - \frac{2y}{20 + t}, \quad y(0) = 100
\]
(b)  (i) \( t_{1/2} = 5600 \) years
\[
\frac{1}{2}N_0 = N_0 e^{kt_{1/2}} \Rightarrow k = \frac{-\ln 2}{t_{1/2}} \approx -0.000124 \text{ year}^{-1}
\]

(ii) \[
0.64N_0 = N(t) = N_0 e^{kt} \Rightarrow t = \frac{\ln 0.64}{k} \approx 3605.6 \text{ years}
\]

(iii) 32%