ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section, (4) your instructors' name, and (5) a grading table. You must work all of the problems on the exam. Unless indicated, show ALL of your work in your bluebook and box in your final answer. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, calculators and ANY electronic devices are NOT permitted. A $8'' \times 11''$, two-sided, sheet of notes is allowed.

TABLE OF LAPLACE TRANSFORMS

$$\mathcal{L}(1) = \frac{1}{s}, \quad \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad \mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad \mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}, \quad \mathcal{L}(\sin(bt)) = \frac{b}{s^2+b^2},$$

$$\mathcal{L}(\cos(bt)) = \frac{s}{s^2+b^2}, \quad \mathcal{L}(e^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2}, \quad \mathcal{L}(e^{at}\cos(bt)) = \frac{s-a}{(s-a)^2+b^2},$$

$$\mathcal{L}(\sinh(bt)) = \frac{b}{s^2-b^2}, \quad \mathcal{L}(\cosh(bt)) = \frac{s}{s^2-b^2}, \quad \mathcal{L}(\text{step}(t-c)) = \frac{e^{-sc}}{s}.$$

1. (30 points) **True/False.** Answer **True** if it is always true, otherwise answer **False**. Write the whole words as opposed to just T or F. No justification is needed and no partial credit will be given.

   (a) (6 points) If $x_1(t)$ is a solution of $x'' + x = f_1(t)$ and $x_2(t)$ is a solution of $x'' + 4x = f_2(t)$, then $x_1(t) + x_2(t)$ is a solution of $2x'' + 5x = f_1(t) + f_2(t)$.

   (b) (6 points) If $f(t)$ and $g(t)$ are linearly dependent functions of t, then the Laplace transforms $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ are linearly dependent functions of s. (You may assume that f and g are defined for all $t \geq 0$ and F and G are defined for all $s > \alpha$, $\alpha \in \mathbb{R}$ constant).

   (c) (6 points) If 1 is an eigenvalue of a 3×3 matrix A, then A is invertible.

   (d) (6 points) The function

   $$f(t) = \begin{cases} 1 & t < 2 \\ e^t & 2 \leq t < 3 \\ 2 & t \geq 3 \end{cases}$$

   may be written as $f(t) = 1 + (e^t - 1) * \text{step}(t-2) + (2 - e^t) * \text{step}(t-3)$, where "step" is the step (or Heaviside) function.

   (e) (6 points) The set of all non-invertible 2×2 matrices is a vector space (with usual matrix operations).

   Solution:

   (a) False. The solution to the homogeneous problem has different eigenvalues and the best guess for the particular solution will depend on the form of the forcing term.

   (b) True. f and g being linearly dependent means there are scalars $c_1, c_2 \in \mathbb{R}$, *not both of them zero*, for which $c_1 f(t) + c_2 g(t) = 0$ for all $t \geq 0$. Taking the Laplace transform of both sides, we see that

   $$\begin{aligned} \mathcal{L}\{c_1 f(t) + c_2 g(t)\} &= \mathcal{L}\{0\} \\ c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\} &= 0 \qquad \mathcal{L} \text{ is linear} \\ c_1 F(s) + c_2 G(s) &= 0 \end{aligned}$$

   which shows that F and G are also linearly dependent since not both of c_1 and c_2 are zero.

   (c) False another eigenvalue could be 0.

   (d) True

   (e) False not closed under addition.

2. (30 points) **Short answer questions**. For the questions in this problem, you do not need to show your work, and no partial credit will be given. If you do submit work, then box your answer, and know that your work will not be graded.

(a) (7 points) Let $\mathbb{P}_4^e = \{$Polynomials $p(x)$ of degree less than or equal to 4 such that $p(x) = p(-x)\}$. What is the dimension of $\mathbb{P}_4^e$?

(b) (8 points) The half-life of the radioactive material *fermium-253* is 3 minutes. Find the time it takes for 3 kg of it to decay to 1 kg.

(c) (7 points) For which values of a does Picard's theorem guarantee the existence of a solution to the initial value problem $y' = |y - t|^{a+3}$, $y(1) = 1$?

(d) (8 points) Find the general solution to $y''' + y = 0$.

Solution:

(a) $\boxed{3}$ Basis $\{1, x^2, x^4\}$.

(b) $\boxed{\frac{3\ln(3)}{\ln(2)}}$ If the amount of fermium remaining is $y(t) = y(0)e^{-kt}$, from the half life we get $k = \ln(3)/2$. Then solving $y(T) = y(0)/3$ we get $T = 3\ln(3)/\ln(2)$.

(c) $\boxed{a \geq -3}$ We need $f(y, t) = |y - t|^{a+3}$ to be continuous in a neighborhood of $(1, 1)$.

(d) $\boxed{y(t) = C_1 e^{-t} + C_2 e^{t/2}\sin(\sqrt{3}t/2) + C_3 e^{t/2}\cos(\sqrt{3}t/2)}$. The characteristic polynomial is $r^3 + 1 = 0$. One root is $r = -1$. Long division gives the other factor polynomial is $1 - r + r^2$, with roots $(1 \pm \sqrt{1-4})/2 = 1/2 \pm i\sqrt{3}/2$.

3. (40 points) A 200 liter tank initially contains 100 liters of pure water. Water enters the tank at a rate of 2 L/hr and the water entering the tank has a kool-aid concentration of 2 gram/L. If a well mixed solution leaves the tank at a rate of 1 L/hr, how much kool-aid powder is dissolved in the tank when it overflows?

Solution: The initial value problem that models this mixing problem is

$$\dot{x} = 4 - \frac{x}{100 + t}; \quad x(0) = 0$$

Students can solve this problem using either the Euler-Lagrange Two-Step Method or the Integrating Factor Method.

The solution is

$$x(t) = \frac{c}{t + 100} + 2\frac{t(t + 200)}{t + 100}$$

and applying the initial condition yields $c = 0$ so that the final solution is

$$x(t) = 2\frac{t(t + 200)}{t + 100}\ .$$

Lastly, the tank will overflow at 100 hours and thus the amount of kool-aid dissolved is

$$x(100) = 2\frac{100(100 + 200)}{100 + 100} = \frac{60000}{200} = 300 \text{ grams of kool} - \text{aid powder}.$$

TURN OVER

4. (40 points) In this problem, use the method of **undetermined coefficients**. Given the second-order, linear differential equation

$$x'' + 2x' + x = f(t),$$

   (a) (10 points) Find the general solution to the homogeneous problem.

   (b) (10 points) If $f(t) = t$, what is the best guess for the particular solution?

   (c) (20 points) If $f(t) = e^{-t}$ and the initial conditions are $x(0) = 1$ and $x'(0) = 0$: (i) What is the best guess for the particular solution? (ii) Solve the initial value problem. (iii) What is the long-term behavior of general solution?

Solution:

   (a) The characteristic equation is $\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$. Therefore, we obtain $\lambda = -1$ with multiplicity 2. The solution to the homogeneous problem is $\boxed{x_h(t) = C_1 e^{-t} + C_2 te^{-t}}$

   (b) (i) $\boxed{x_p(t) = At + B}$.

   (c) (i) $\boxed{x_p(t) = At^2 e^{-t}}$. (ii) $\boxed{x_p(t) = e^{-t} + te^{-t} + (1/2)t^2 e^{-t}}$. (iii) The exponential decay is faster than the parabolic growth, so the solution decays towards 0.

5. (40 points) The following questions (a), (b), (c) are not related:

   (a) (10 points) Let a and r be real numbers with $a \geq 0$. Use the Wronskian to determine all values of both a and r for which $S = \{\sin(rt), t^a \sin(rt)\}$, defined for $t > 0$, is a linearly *dependent* set of functions.

   (b) i. (8 points) Is the vector $\begin{bmatrix} 8 \\ -16 \\ 12 \end{bmatrix}$ in the span of

$$U = \left\{ \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -8 \\ -3 \end{bmatrix} \right\}?$$

   Be sure to justify your answer.

   ii. (8 points) Do the vectors in U form a basis of $\mathbb{R}^3$? Why or why not?

   (c) Here is a matrix A and its RREF (you do *not* have to verify this row reduction! Just use this information to answer the following questions):

$$\begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   i. (7 points) Find a basis for the span of the column vectors of A.

   ii. (7 points) Let $\mathbb{W}$ be the solution space of the equation $A\mathbf{x} = \mathbf{0}$. Find a basis for $\mathbb{W}$.

Solution:

   (a) Note that

$$\begin{aligned} W[\sin(rt), t^a \sin(rt)](t) &= \begin{vmatrix} \sin(rt) & t^a \sin(rt) \\ r\cos(rt) & at^{a-1}\sin(rt) + rt^a \cos(rt) \end{vmatrix} \\ &= at^{a-1}\sin^2(rt) + rt^a \sin(rt)\cos(rt) - rt^a \sin(rt)\cos(rt) \\ &= at^{a-1}\sin^2(rt). \end{aligned}$$

   This suggests we consider $a = 0$ or $r = 0$. If $a = 0$ the two original functions are the same (they're both $\sin(rt)$) and so linearly dependent. If $r = 0$ both original functions are the constant function zero and so linearly dependent. So if $a \neq 0$ and $r \neq 0$, the Wronskian above is *not* identically zero for all t (take $t = \pi/2r$ for instance) and so the two given functions are linearly independent. This means the given functions are linearly dependent only when $a = 0$ or $r = 0$.

(b) i. To try and solve $c_1 \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ -8 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ -16 \\ 12 \end{bmatrix}$ is the same as solving

$$\begin{bmatrix} 2 & 2 & 4 \\ -4 & -4 & -8 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -16 \\ 12 \end{bmatrix}$$

so we row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so this system has infinitely many solutions actually. So $\boxed{YES}$.

ii. $\boxed{NO}$. You can see from the row reduction in the previous part that the vectors in U are linearly *dependent* and so cannot form a basis.

(c) i. From the RREF the first, second and fourth column vectors of A are linearly independent, so a basis of the span of the column vectors of A is

$$\left\{ \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \right\}$$

ii. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ be a solution to $A\mathbf{x} = \mathbf{0}$. From the RREF we know that

$$\begin{aligned} x_1 + \frac{4}{3}x_5 = 0 &\implies x_1 = -\frac{4}{3}x_5 \\ x_2 + 3x_3 - 2x_5 = 0 &\implies x_2 = -3x_3 + 2x_5 \\ x_4 - 2x_5 = 0 &\implies x_4 = 2x_5, \end{aligned}$$

x_3 and x_5 are free. So set $x_3 = s$, $x_5 = t$ and we can write

$$\mathbf{x} = \begin{bmatrix} -\frac{4}{3}t \\ -3s + 2t \\ s \\ 2t \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{4}{3} \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

and so a basis for W is

$$\left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{4}{3} \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

6. (40 points) Consider the matrix

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$$

(a) (10 points) Compute all eigenvalues of A.

(b) (15 points) For each eigenvalue, find a corresponding eigenvector of A.

(c) (15 points) Find the general solution to the system of differential equations $\vec{x}' = A\vec{x}$. Write the solution in terms of real numbers only.

Solution:

(a) The charactersitic equation of A is

$$\begin{aligned} p(\lambda) &= det(\lambda I - A) \\ &= (\lambda - 1)(\lambda + 1) - (-5) \\ &= \lambda^2 + 4 \end{aligned}$$

So the eigenvalues are $\lambda_1 = 2i$ and $\lambda_2 = -2i$.

(b) To find an eigenvector for the eigenvalue $\lambda_1 = 2i$, we solve

$$(A - \lambda_1 I)\vec{v} = \begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \vec{v} = 0$$

We now do the Gauss-Jordan via row operations:
$\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \xrightarrow{R_1^* = \frac{1+2i}{5}R_1} = \begin{bmatrix} 1 & -1-2i \\ 1 & -1-2i \end{bmatrix} \xrightarrow{R*_2 = R_2 - R_1} = \begin{bmatrix} 1 & -1-2i \\ 0 & 0 \end{bmatrix}$ x_2 is the only free variable, set $x_2 = 1$, we solve the homogeneous equations to get: $x_1 = 1 + 2i$.

Thus an eigenvector for $\lambda_1 = 2i$ is $\vec{v}_1 = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$.

An eigenvector for the eigenvalue $\lambda_2 = -2i$ is $\vec{v}_2 = \overline{\vec{v}_1} = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$.

(c) To find the general solutions of the system. $\lambda_1 = 2i$ and an eigenvector belonging to $\lambda_1 = 2i$

$$\vec{v_1} = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Hence a fundamental solutions is given by

$$\vec{x}_{re}(t) = e^{0t} cos2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{0t} sin2t \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} cos2t - 2sin2t \\ cos2t \end{bmatrix}$$

$$\vec{x}_{Im}(t) = e^{0t} sin2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{0t} cos2t \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} sin2t + 2cos2t \\ sin2t \end{bmatrix}$$

The general solution is therefore

$$\vec{x}(t) = c_1 \begin{bmatrix} cos2t - 2sin2t \\ cos2t \end{bmatrix} + c_2 \begin{bmatrix} sin2t + 2cos2t \\ sin2t \end{bmatrix}$$

7. (30 points) Consider the following initial value problem: $x'' + x = 10\sin(2t)$, $x(0) = 0$ and $x'(0) = 0$.

(a) (7 points) Find the Laplace Transform of the differential equation.

(b) (8 points) Solve for the solution of the differential equation in the s-domain (i.e., find $X(s)$).

(c) (15 points) Solve for the solution of the differential equation in the t-domain (i.e., find $x(t)$).

Solution:

(a) Taking the Laplace Transform of both sides of the DE gives $s^2X(s) - sx(0) - x'(0) + X(s) = 10\frac{2}{s^2+4}$, which simplifies to $s^2X(s) + X(s) = 10\frac{2}{s^2+4}$ after substituting the initial conditions.

(b) $X(s) = \frac{20}{(s^2+1)(s^2+4)}$

(c) Partial Fractions gives $X(s) = \frac{20/3}{s^2+1} - \frac{20/3}{s^2+4} = \frac{20}{3}\frac{1}{s^2+1} - \frac{10}{3}\frac{2}{s^2+4}$ so $x(t) = \frac{20}{3}\sin(t) - \frac{10}{3}\sin(2t)$.

IF YOU ARE FINISHED AND HAVE TIME, CHECK YOUR ANSWERS.