ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section, (4) your instructors' name, and (5) a grading table. You must work all of the problems on the exam. Unless indicated, show ALL of your work in your bluebook and box in your final answer. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, calculators and ANY electronic devices are NOT permitted. A $8'' \times 11''$, two-sided, sheet of notes is allowed.

TABLE OF LAPLACE TRANSFORMS

$$\mathcal{L}(1) = \frac{1}{s}, \quad \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad \mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad \mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}, \quad \mathcal{L}(\sin(bt)) = \frac{b}{s^2+b^2},$$

$$\mathcal{L}(\cos(bt)) = \frac{s}{s^2+b^2}, \quad \mathcal{L}(e^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2}, \quad \mathcal{L}(e^{at}\cos(bt)) = \frac{s-a}{(s-a)^2+b^2},$$

$$\mathcal{L}(\sinh(bt)) = \frac{b}{s^2-b^2}, \quad \mathcal{L}(\cosh(bt)) = \frac{s}{s^2-b^2}, \quad \mathcal{L}(\text{step}(t-c)) = \frac{e^{-sc}}{s}.$$

1. (30 points) **True/False.** Answer **True** if it is always true, otherwise answer **False**. Write the whole words as opposed to just T or F. No justification is needed and no partial credit will be given.

   (a) (6 points) If $x_1(t)$ is a solution of $x'' + x = f_1(t)$ and $x_2(t)$ is a solution of $x'' + 4x = f_2(t)$, then $x_1(t) + x_2(t)$ is a solution of $2x'' + 5x = f_1(t) + f_2(t)$.

   (b) (6 points) If $f(t)$ and $g(t)$ are linearly dependent functions of t, then the Laplace transforms $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ are linearly dependent functions of s. (You may assume that f and g are defined for all $t \geq 0$ and F and G are defined for all $s > \alpha$, $\alpha \in \mathbb{R}$ constant).

   (c) (6 points) If 1 is an eigenvalue of a 3×3 matrix A, then A is invertible.

   (d) (6 points) The function

   $$f(t) = \begin{cases} 1 & t < 2 \\ e^t & 2 \leq t < 3 \\ 2 & t \geq 3 \end{cases}$$

   may be written as $f(t) = 1 + (e^t - 1) * \text{step}(t-2) + (2 - e^t) * \text{step}(t-3)$, where "step" is the step (or Heaviside) function.

   (e) (6 points) The set of all non-invertible 2×2 matrices is a vector space (with usual matrix operations).

2. (30 points) **Short answer questions**. For the questions in this problem, you do not need to show your work, and no partial credit will be given. If you do submit work, then box your answer, and know that your work will not be graded.

   (a) (7 points) Let $\mathbb{P}_4^e = \{\text{Polynomials } p(x) \text{ of degree less than or equal to 4 such that } p(x) = p(-x)\}$. What is the dimension of $\mathbb{P}_4^e$?

   (b) (8 points) The half-life of the radioactive material *fermium-253* is 3 minutes. Find the time it takes for 3 kg of it to decay to 1 kg.

   (c) (7 points) For which values of a does Picard's theorem guarantee the existence of a solution to the initial value problem $y' = |y - t|^{a+3}$, $y(1) = 1$?

   (d) (8 points) Find the general solution to $y''' + y = 0$.

3. (40 points) A 200 liter tank initially contains 100 liters of pure water. Water enters the tank at a rate of 2 L/hr and the water entering the tank has a kool-aid concentration of 2 gram/L. If a well mixed solution leaves the tank at a rate of 1 L/hr, how much kool-aid powder is dissolved in the tank when it overflows?

TURN OVER

4. (40 points) In this problem, use the method of **undetermined coefficients**. Given the second-order, linear differential equation

$$x'' + 2x' + x = f(t),$$

   (a) (10 points) Find the general solution to the homogeneous problem.

   (b) (10 points) If $f(t) = t$, what is the best guess for the particular solution?

   (c) (20 points) If $f(t) = e^{-t}$ and the initial conditions are $x(0) = 1$ and $x'(0) = 0$: (i) What is the best guess for the particular solution? (ii) Solve the initial value problem. (iii) What is the long-term behavior of general solution?

5. (40 points) The following questions (a), (b), (c) are not related:

   (a) (10 points) Let a and r be real numbers with $a \geq 0$. Use the Wronskian to determine all values of both a and r for which $S = \{\sin(rt), t^a \sin(rt)\}$, defined for $t > 0$, is a linearly *dependent* set of functions.

   (b) i. (8 points) Is the vector $\begin{bmatrix} 8 \\ -16 \\ 12 \end{bmatrix}$ in the span of

$$U = \left\{ \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -8 \\ -3 \end{bmatrix} \right\}?$$

   Be sure to justify your answer.

   ii. (8 points) Do the vectors in U form a basis of $\mathbb{R}^3$? Why or why not?

   (c) Here is a matrix A and its RREF (you do *not* have to verify this row reduction! Just use this information to answer the following questions):

$$\begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   i. (7 points) Find a basis for the span of the column vectors of A.

   ii. (7 points) Let $\mathbb{W}$ be the solution space of the equation $A\mathbf{x} = \mathbf{0}$. Find a basis for $\mathbb{W}$.

6. (40 points) Consider the matrix

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$$

   (a) (10 points) Compute all eigenvalues of A.

   (b) (15 points) For each eigenvalue, find a corresponding eigenvector of A.

   (c) (15 points) Find the general solution to the system of differential equations $\vec{x}' = A\vec{x}$. Write the solution in terms of real numbers only.

7. (30 points) Consider the following initial value problem: $x'' + x = 10\sin(2t)$, $x(0) = 0$ and $x'(0) = 0$.

   (a) (7 points) Find the Laplace Transform of the differential equation.

   (b) (8 points) Solve for the solution of the differential equation in the s-domain (i.e., find $X(s)$).

   (c) (15 points) Solve for the solution of the differential equation in the t-domain (i.e., find $x(t)$).

IF YOU ARE FINISHED AND HAVE TIME, CHECK YOUR ANSWERS.