APPM 2360: Midterm exam 3
April 18, 2018

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A letter sized one page crib sheet is allowed.

Solution: APPM 2360 **Exam 3** **Spring 2018**

Laplace transform table

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad \mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}$$

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2} \qquad \mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2} \qquad \mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$$

$$\mathcal{L}\{\sinh(bt)\} = \frac{b}{s^2-b^2} \qquad \mathcal{L}\{\cosh(bt)\} = \frac{s}{s^2-b^2}$$

Problem 1: (30 points, 6 points each) **True/False** (answer True if it is always true otherwise answer False).

(a) If λ is an eigenvalue of an $n \times n$ matrix A, then $1/\lambda$ is an eigenvalue of A^T.
(b) The set of eigenvectors belonging to a particular eigenvalue λ of an $n \times n$ matrix A, together with the zero vector, is a subspace of $\mathbb{R}^n$.
(c) Resonance in a mass-spring system can be prevented by changing the initial velocity of the mass.
(d) The Laplace transform of $f(t) = e^t + e^{-t}$ is $F(s) = \frac{s}{s^2-1}$.
(e) In the method of Undetermined Coefficients, the form of the particular solution one should guess for the equation $y'' + 2y' - 3y = t^2 + \sin(t)$ is $y_p(t) = At^2 + Bt + C + D\sin(t) + E\cos(t)$.

Solution:

(a) $\boxed{FALSE}$ In general A and A^T have the *same* eigenvalues, so for $n \times n$ matrices that have an eigenvalues $\lambda \neq 1$, this will be false. Specifically let $A = 5I_2$. Then A has the single eigenvalue $\lambda = 5$. But $A^T = A$ so A^T has the single eigenvalue $\lambda = 5$ as well, *not* $1/5$.
(b) $\boxed{TRUE}$. Use the Subspace Theorem to prove this.
(c) $\boxed{FALSE}$. Resonance occurs in an undamped mass-spring system when the frequency $\omega_0 = \sqrt{k/m}$ matches the frequency ω_f of an outside sinusoidal forcing function, and those frequencies being equal is not affected by the initial velocity (or positon) of the object.
(d) $\boxed{FALSE}$. $F(s) = \frac{1}{s-1} + \frac{1}{s+1} = \frac{2s}{s^2-1}$, $s > 1$.
(e) $\boxed{TRUE}$. The roots of the characteristic equation are $r = -3$ and $r = 1$, so none of the terms considered are a solution of the homogeneous equation.

Problem 2: (30 points, 10 points each) **Short Answer** for the following problems. No justification is needed and no partial credit is given.

(a) Compute the inverse Laplace transform of

$$F(s) = \frac{s}{s^2 + 4s + 8}.$$

(b) Let A be a 2×2 matrix. The eigenvalues of A are $\lambda_1 = 13$ and $\lambda_2 = 31$. What are the eigenvalues of AA^{-1}?

(c) Find a basis for the solution space of $x'' - x = 0$.

Solution:

(a) Completing squares in the denominator, the fraction can be rewritten as

$$\frac{s}{s^2+4s+8} = \frac{s}{(s+2)^2+4} = \frac{s+2}{(s+2)^2+2^2} - \frac{2}{(s+2)^2+2^2}.$$

From the table, $\boxed{\mathcal{L}^{-1}\{F(s)\} = e^{-2t}[\cos(2t) - \sin(2t)]}$.

(b) As $AA^{-1} = \mathbb{I}$, $\boxed{\text{the only eigenvalue of } AA^{-1} \text{ is } 1}$ (with algebraic multiplicity 2).

(c) A basis for the solution space is $\boxed{\{e^t, e^{-t}\}}$.

Problem 3: (30 points) A harmonic oscillator is described by the differential equation

$$x'' + bx' + 4x = f(t)$$

where $f(t)$ is a forcing function and b is a tunable parameter. Answer the following questions:

(a) (10 points) If $f(t) = 0$ and $b = 0$, (i) find the general solution to the homogeneous problem. (ii) What is the the period of oscillation?

(b) (10 points) Calculate the value of b such that the harmonic oscillator is critically damped.

(c) (10 points) If $f(t) = F_0\cos(\omega_f t)$, the particular solution can be written as $x_p(t) = A\cos(\omega_f t - \delta)$. Give the values or conditions for ω_f such that $|A| \to \infty$ as $b \to 0$.

Solution:

(a) (i) Solving the characteristic equation, we obtain $\lambda_\pm = \pm i\omega_0 = \pm 2i$. Therefore, $\boxed{x(t) = C_1 e^{2it} + C_2 e^{-2it}}$ or $\boxed{C\cos(2t + \phi)}$. (ii) The period of oscillation is $\boxed{T = 2\pi/\omega_0 = \pi \text{ sec}}$.

(b) A critically damped solution is obtained when we have repeated real roots. For this, we need to ensure that $b^2 - 4\omega^2 = 0$. We obtain $\boxed{b = 4}$.

(c) The limit $b \to 0$ corresponds to the undamped case, at which resonance occurs at $\omega_f = \omega_0 = \boxed{2 \text{ rad/sec}}$.

Problem 4: (30 points) The following questions are unrelated.

(a) (12 points) i) Is $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ an eigenvector for $\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$?

ii) If so, what is the corresponding eigenvalue?

(b) (6 points) Find all eigenvalues of $\begin{bmatrix} 1 & -4 \\ 4 & 11 \end{bmatrix}$.

(c) (12 points) One eigenvalue of $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$ is $\lambda_1 = 5$.

i) Find the corresponding eigenvector $\vec{v_1}$.

ii) Another eigenvector of this matrix is $\vec{v_2} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. Is $10\vec{v_2}$ also an eigenvector?

Solution:

(a) i) $\boxed{\text{Yes}}$: $\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

ii) The corresponding eigenvalue is $\boxed{2}$

(b) Solve $\det \begin{bmatrix} 1-\lambda & -4 \\ 4 & 11-\lambda \end{bmatrix} = 0$, giving $\lambda^2 - 12\lambda + 27 = 0$, so $\boxed{\lambda = 3, 9}$

(c) i) Solve $A - 5I = \vec{0}$ by row reducing $A - 5I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -5 & 3 \\ 2 & 3 & -5 \end{bmatrix}$ which gives $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

with corresponding solution $\boxed{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$ or any scalar multiple

ii) $\boxed{\text{Yes}}$

Problem 5: (30 points) Consider the differential equation $y'' - 2y' - 3y = f(t)$.

(a) Applying the method of undetermined coefficients, write down the FORM of the particular solution for (**do not compute the coefficients**):

(i) (8 points) $f(t) = t^2 e^{-t}$,

(ii) (8 points) $f(t) = e^{2t} \cos(2t)$.

(b) (14 points) Find the particular solution for $f(t) = -20\cos(t)$ using the method of undetermined coefficients.

Solution:

(a) (i) $y_p(t) = t(At^2 + Bt + C)e^{-t}$

(ii) $y_p(t) = e^{2t}(A\cos(2t) + B\sin(2t))$

(b) $y = A\cos(t) + B\sin(t)$; $y'' - 2y' - 3y = (-2B - 4A)\cos(t) + 2(A - 2B)\sin(t) = -20\cos(t)$

$\Rightarrow A - 2B = 0;\ -2B - 4A = -20$ and we obtain $A = 4, B = 2$.

$\boxed{y_p(t) = 4\cos(t) + 2\sin(t)}$