Problem 1: (30 points, 6 points each) True/False (Answer True if it is always true otherwise answer False).

(a) Let $U_{3 \times 3} = \{ 3 \times 3$ upper triangular matrices with real entries} with the usual matrix operations. The dimension of the space $U_{3 \times 3}$ is 9.

(b) If a $n \times n$ matrix $A$ is noninvertible, the system $A\vec{x} = \vec{b}$ has no solutions.

(c) The polynomials $p_1(t) = t + 1$, $p_2(t) = t - 1$, $p_3(t) = 1 + t + t^2$ are linearly independent.

(d) The set $V = \{ 2 \times 2$ matrices with zero determinant $\}$, with usual matrix operations, is a vector space.

(e) For any $n \times n$ matrix $A$, $|AA^T| = |A^T A|$.

Solution:

(a) False. It is 6.

(b) False. It could have infinite solutions.

(c) True. The Wronskian gives $W(t) = 4 \neq 0$.

(d) False. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Both have zero determinant, but their sum doesn’t. Therefore the set is not closed under addition.

(e) True: Since $|A| = |A^T|$, $|AA^T| = |A||A^T| = |A|^2$ and $|A^T A| = |A^T||A| = |A|^2$.

Problem 2: (30 points, 10 points each) Short Answer for the following problems. No justification is needed.

(a) Calculate all equilibria of the system of differential equations

\[ \begin{align*}
  x' &= \frac{y^2}{2} - x - 4, \\
  y' &= y - x.
\end{align*} \]

(b) Given $B = P^{-1}A$, where $A, B, P$ are all invertible matrices, find an expression for (i) $|B|$ and (ii) $B^{-1}$ in terms of $A$ and $P$.

(c) What is a basis for Span \( \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\} \)?

Solution:

(a) The equilibria are obtained where the nullclines intercept. We look for solutions of $y^2/2 = x + 4$ and $y = x$. Replacing $y$, we obtain $x^2/2 - x - 4 = 0$. Solving, the equilibria are $(-2, -2)$ and $(4, 4)$.

(b) (1) From properties of determinants, $|P^{-1}| = 1/|P|$. Then, $|B| = |A|$. (2) $B^{-1} = (P^{-1}A)^{-1} = P^{-1}A^{-1}P$.

(c) Reducing the matrix of column vectors we obtain

\[
\begin{bmatrix}
2 & 6 & 1 \\
2 & 3 & 2 \\
2 & -3 & 4
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 3/2 \\
0 & 1 & -1/3 \\
0 & 0 & 0
\end{bmatrix},
\]
which is rank 2, with the pivot columns being the first and second columns. Thus the basis for the span is the first two columns of the span:

\[
\{ \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix} \}\]

**Problem 3:** (30 points) Consider the linear system of equations \(Ax = b\), where

\[
A = \begin{bmatrix} 3 & k & 0 \\ 0 & 1 & 5 \\ 0 & k & k \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ k \\ 0 \end{bmatrix}
\]

(a) (20 points) Solve this system when \(k = 0\) using elementary row operations.
(b) (10 points) For which values of \(k\) does this system have (i) a unique solution, (ii) no solution, (iii) an infinite number of solutions.

**Solution:**

(a) One row operation gives the following augmented matrix:

\[
\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

Thus, the solution is \(x = \begin{bmatrix} 1/3 \\ -5x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}\)

(b) One row operation gives the following augmented matrix:

\[
\begin{bmatrix} 3 & k & 0 & 1 \\ 0 & 1 & 5 & k \\ 0 & 0 & -4 & -k^2 \end{bmatrix}
\]

(i) \(k \neq 0\), (ii) there is always a solution, and (iii) \(k = 0\)

**Problem 4:** (30 points) In this problem \(A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & k \\ 0 & 1 & 0 \end{bmatrix}\)

(a) (10 points) Compute the determinant of \(A\).
(b) (5 points) Find all possible value(s) of \(k\) for which \(A\) is not invertible.
(c) (15 points) Compute \(A^{-1}\), the inverse of \(A\), for \(k = 2\).

**Solution:**

\[
\det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & k \\ 0 & 1 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & k - 2 \\ 0 & 1 & 1 \end{bmatrix} = 1 \cdot (1 - (k - 2)) = 3 - k
\]

b \(\det(A) = 3 - k = 0 \Rightarrow k = 3\)

c For \(k = 2\), \([A \ I]=
\[
\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{pmatrix}
\]

\[
\rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{pmatrix}
\]

Hence \(A^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}\)

**Problem 5:** (30 points, 10 points each)
(a) For what value(s) of $a$, if any, can you conclude that the set
\[ \{ \sin(at), \cos(at), 1 \} \]
is linearly independent?

(b) Let $\mathbb{V} = M_{22} = \{2 \times 2 \text{ matrices with real entries}\}$ and $\mathbb{W} = \{A \in \mathbb{V} \text{ such that } A^T = -A\}$. Is $\mathbb{W}$ a vector subspace of $\mathbb{V}$? Be sure to fully justify your answer.

(c) Consider the following system of linear equations
\[ \begin{align*}
x_1 + 2x_2 + x_3 - x_4 &= 0 \\
x_3 + x_4 &= 0.
\end{align*} \]
(i) Find a basis for the solution space.
(ii) What is the dimension of the solution space?

Solution:

(a) Consider the Wronskian
\[
W[\sin(at), \cos(at), 1](t) = \begin{vmatrix}
\sin(at) & \cos(at) & 1 \\
a \cos(at) & -a \sin(at) & 0 \\
-a^2 \sin(at) & -a^2 \cos(at) & 0
\end{vmatrix}
= -a^3 \cos^2(at) - a^3 \sin^2(at)
= -a^3(\cos^2(at) + \sin^2(at)) = -a^3
\]
We can conclude that the functions are linearly independent for $\boxed{a \neq 0}$.

(b) $\boxed{\mathbb{W}}$ is a subspace of $\mathbb{V}$. Consider any $A, B \in \mathbb{W}$. Thus we know that $A^T = -A$ and $B^T = -B$.

Now let $C = \alpha A + \beta B$ for $\alpha, \beta, \in \mathbb{R}$.

Consider
\[
C^T = (\alpha A + \beta B)^T
= \alpha A^T + \beta B^T
= - (\alpha A + \beta A)
= -C
\]
Therefore, $C \in \mathbb{W}$ and thus $\mathbb{W}$ is a subspace of $\mathbb{V}$.

(c) First convert to the augmented matrix and row reduce:
\[
\begin{bmatrix}
1 & 2 & 1 & -1 & 0 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 0 & -2 & 0 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix}
\]
Thus, the $w$ and $y$ columns are pivot columns and the $x$ and $z$ columns are free variables. Let $x = t$ and $z = s$ and thus
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
-2t + 2s \\
t \\
-s \\
s
\end{bmatrix} = t \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix} + s \begin{bmatrix}
2 \\
0 \\
-1 \\
1
\end{bmatrix}.
\]
Thus (i) a basis is
\[
\left\{ \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
2 \\
0 \\
-1 \\
1
\end{bmatrix} \right\}
\]
and (ii) the dimension of the solution space is $\boxed{2}$. 