APPM 2360: Midterm exam 2
March 14, 2018

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your recitation section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A letter sized one page crib sheet is allowed.

Solution: APPM 2360 **Exam 2** **Spring 2018**

Problem 1: (30 points, 6 points each) **True/False** (Answer True if it is always true otherwise answer False).

(a) Let $\mathbb{U}_{3\times3} = \{3 \times 3$ upper triangular matrices with real entries$\}$ with the usual matrix operations. The dimension of the space $\mathbb{U}_{3\times3}$ is 9.
(b) If a $n \times n$ matrix A is noninvertible, the system $A\vec{x} = \vec{b}$ has no solutions.
(c) The polynomials $p_1(t) = t + 1$, $p_2(t) = t - 1$, $p_3(t) = 1 + t + t^2$ are linearly independent.
(d) The set $\mathbb{V} = \{2 \times 2$ matrices with zero determinant$\}$, with usual matrix operations, is a vector space.
(e) For any $n \times n$ matrix A, $|AA^T| = |A^TA|$.

Solution:

(a) $\boxed{False}$ It is 6.
(b) $\boxed{False}$ It could have infinite solutions.
(c) $\boxed{True}$ The Wronskian gives $W(t) = 4 \neq 0$.
(d) $\boxed{False}$ Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Both have zero determinant, but their sum doesn't. Therefore the set is not closed under addition.
(e) $\boxed{True:}$ Since $|A| = |A^T|$, $|AA^T| = |A||A^T| = |A|^2$ and $|A^TA| = |A^T||A| = |A|^2$.

Problem 2: (30 points, 10 points each) **Short Answer** for the following problems. No justification is needed.

(a) Calculate all equilibria of the system of differential equations

$$\begin{aligned} x' &= \frac{y^2}{2} - x - 4, \\ y' &= y - x. \end{aligned}$$

(b) Given $B = P^{-1}AP$, where A, B, P are all invertible matrices, find an expression for (i) $|B|$ and (ii) B^{-1} in terms of A and P.
(c) What is a basis for Span$\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}$?

Solution:

(a) The equilibria are obtained where the nullclines intercept. We look for solutions of $y^2/2 = x + 4$ and $y = x$. Replacing y, we obtain $x^2/2 - x - 4 = 0$. Solving, the equilibria are $(-2, -2)$ and $(4, 4)$
(b) (1) From properties of determinants, $|P^{-1}| = 1/|P|$. Then, $|B| = |A|$. (2) $B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}P$.
(c) Reducing the matrix of column vectors we obtain

$$\begin{bmatrix} 2 & 6 & 1 \\ 2 & 3 & 2 \\ 2 & -3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix},$$

which is rank 2, with the pivot columns being the first and second colums. Thus the basis for the span is the first two columns of the span:

$$\boxed{\left\{\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix}\right\}}$$

Problem 3: (30 points) question Consider the linear system of equations $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 3 & k & 0 \\ 0 & 1 & 5 \\ 0 & k & k \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ k \\ 0 \end{bmatrix}$$

(a) (20 points) Solve this system when $k = 0$ using elementary row operations.
(b) (10 points) For which values of k does this system have (i) a unique solution, (ii) no solution, (iii) an infinite number of solutions.

Solution:

(a) One row operation gives the following augmented matrix: $\left[\begin{array}{ccc|c} 3 & 0 & 0 & 1 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Thus, the solution is $\mathbf{x} = \begin{bmatrix} 1/3 \\ -5x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}$

(b) One row operation gives the following augmented matrix: $\left[\begin{array}{ccc|c} 3 & k & 0 & 1 \\ 0 & 1 & 5 & k \\ 0 & 0 & -4k & -k^2 \end{array}\right]$

(i) $k \neq 0$, (ii) there is always a solution, and (iii) $k = 0$

Problem 4: (30 points) In this problem $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & k \\ 0 & 1 & 1 \end{bmatrix}$

(a) (10 points) Compute the determinant of A.
(b) (5 points) Find all possible value(s) of k for which A **is not invertible**.
(c) (15 points) Compute A^{-1}, the inverse of A, for $k = 2$.

Solution:

a $det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & k \\ 0 & 1 & 1 \end{bmatrix} = det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & k-2 \\ 0 & 1 & 1 \end{bmatrix} = 1 * (1 - (k-2)) = \boxed{3 - k}$

b $Det(A) = 3 - k = 0 \Rightarrow \boxed{k = 3}$.

c For $k = 2$, $[A \;\; I]$=

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right).$$

Hence $A^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$

Problem 5: (30 points, 10 points each)

(a) For what value(s) of a, if any, can you conclude that the set

$$\{ \sin(at) \ , \ \cos(at) \ , \ 1 \}$$

is linearly independent?

(b) Let $\mathbb{V} = \mathbb{M}_{22} = \{2 \times 2 \text{ matrices with real entries}\}$ and $\mathbb{W} = \{A \in \mathbb{V} \text{ such that } A^T = -A\}$. Is $\mathbb{W}$ a vector subspace of $\mathbb{V}$? Be sure to fully justify your answer.

(c) Consider the following system of linear equations

$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 &= 0 \\ x_3 + x_4 &= 0. \end{aligned}$$

(i) Find a basis for the solution space.

(ii) What is the dimension of the solution space?

Solution:

(a) Consider the Wronskian

$$\begin{aligned} W[\sin(at), \cos(at), 1](t) &= \left\| \begin{bmatrix} \sin(at) & \cos(at) & 1 \\ a\cos(at) & -a\sin(at) & 0 \\ -a^2\sin(at) & -a^2\cos(at) & 0 \end{bmatrix} \right\| \\ &= -a^3\cos^2(at) - a^3\sin^2(at) \\ &= -a^3(\cos^2(at) + \sin^2(at)) = -a^3 \end{aligned}$$

We can conclude that the functions are linearly independent for $\boxed{a \neq 0}$.

(b) $\boxed{Yes}$, $\mathbb{W}$ is a subspace of $\mathbb{V}$. Consider any $A, B \in \mathbb{W}$. Thus we know that $A^T = -A$ and $B^T = -B$. Now let $C = \alpha A + \beta B$ for $\alpha, \beta, \in \mathbb{R}$.

Consider

$$\begin{aligned} C^T &= (\alpha A + \beta B)^T \\ &= \alpha A^T + \beta B^T \\ &= -(\alpha A + \beta A) \\ &= -C \end{aligned}$$

Therefore, $C \in \mathbb{W}$ and thus $\mathbb{W}$ is a subspace of $\mathbb{V}$.

(c) First convert to the augmented matrix and row reduce:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Thus, the w and y columns are pivot colums and the x and z colums are free variables. Let $x = t$ and $z = s$ and thus

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t + 2s \\ t \\ -s \\ s \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

Thus (i) a basis is

$$\boxed{\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}}$$

and (ii) the dimension of the solution space is $\boxed{2}$.