Problem 1: (30 points, 6 points each) True/False (Answer True if it is always true otherwise answer False).
(a) Let $U_{3 \times 3} = \{3 \times 3$ upper triangular matrices with real entries$\}$ with the usual matrix operations. The dimension of the space $U_{3 \times 3}$ is 9.
(b) If a $n \times n$ matrix $A$ is noninvertible, the system $A\vec{x} = \vec{b}$ has no solutions.
(c) The polynomials $p_1(t) = t + 1$, $p_2(t) = t - 1$, $p_3(t) = 1 + t + t^2$ are linearly independent.
(d) The set $V = \{2 \times 2$ matrices with zero determinant$\}$, with usual matrix operations, is a vector space.
(e) For any $n \times n$ matrix $A$, $|AA^T| = |A^TA|$.

Problem 2: (30 points, 10 points each) Short Answer for the following problems. No justification is needed.
(a) Calculate all equilibria of the system of differential equations
\[
\begin{align*}
x' &= \frac{y^2}{2} - x - 4, \\
y' &= y - x.
\end{align*}
\]
(b) Given $B = P^{-1}AP$, where $A, B, P$ are all invertible matrices, find an expression for (i) $|B|$ and (ii) $B^{-1}$ in terms of $A$ and $P$.
(c) What is a basis for $\text{Span}\left\{\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}\right\}$?

Problem 3: (30 points) question Consider the linear system of equations $Ax = b$, where
\[
A = \begin{bmatrix}
3 & k & 0 \\
0 & 1 & 5 \\
0 & k & k
\end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ k \\ 0 \end{bmatrix}
\]
(a) (20 points) Solve this system when $k = 0$ using elementary row operations.
(b) (10 points) For which values of $k$ does this system have (i) a unique solution, (ii) no solution, (iii) an infinite number of solutions.

Problem 4: (30 points) In this problem $A = \begin{bmatrix} 1 & 0 & 1 \\
2 & 1 & k \\
0 & 1 & 1 \end{bmatrix}$
(a) (10 points) Compute the determinant of $A$.
(b) (5 points) Find all possible value(s) of $k$ for which $A$ is not invertible.
(c) (15 points) Compute $A^{-1}$, the inverse of $A$, for $k = 2$.

Problem 5: (30 points, 10 points each)
(a) For what value(s) of $a$, if any, can you conclude that the set
\[
\{ \sin(at), \cos(at), 1 \}
\]
is linearly independent?
(b) Let $\mathbb{V} = \mathbb{M}_{2 \times 2} = \{2 \times 2$ matrices with real entries$\}$ and $\mathbb{W} = \{A \in \mathbb{V}$ such that $A^T = -A\}$. Is $\mathbb{W}$ a vector subspace of $\mathbb{V}$? Be sure to fully justify your answer.
(c) Consider the following system of linear equations
\[
\begin{align*}
x_1 + 2x_2 + x_3 - x_4 &= 0 \\
x_3 + x_4 &= 0.
\end{align*}
\]
(i) Find a basis for the solution space.
(ii) What is the dimension of the solution space?