APPM 2360: Midterm exam 1
February 14, 2018

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your recitation section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A letter sized one page crib sheet is allowed.

Solution: APPM 2360 **Exam1** **Spring 2018**

Problem 1: (30 points, 6 points each) **True/False** (answer True if it is always true, otherwise answer False).

(a) $y'' + y'y + ty = 0$ is a second order linear homogeneous differential equation.
(b) The operator $L[y] = y' + 3y + 1$ satisfies the two properties of linear operators.
(c) If $y_1(t)$ and $y_2(t)$ are two solutions of the differential equation $y^2 - y' + y = 0$, then for any constants c_1 and c_2, $c_1y_1(t) + c_2y_2(t)$ is also a solution.
(d) The conditions of Picard's Theorem about **both** the existence and the uniqueness are satisfied by the IVP $y' = (ty)^{1/3}$ with $y(0) = 1$.
(e) The equilibrium solution $y = 3$ to the differential equation $y' = -2(y-1)(y-3)$ is stable.

Solution:

(a) $\boxed{False:}$ It is not linear.
(b) $\boxed{False:}$ It is not linear operator because $L[2y] \neq 2L[y]$.
(c) $\boxed{False:}$ The given diffeq is nonlinear, so the superposition principle isn't guaranteed to hold. In fact, $y_1(t) = -1$ is a solution but $2y_1(t)$ is not. You could also use $y_2(t) = \dfrac{e^t}{1-e^t}$.
(d) $\boxed{True:}$ Setting $f(t,y) = (ty)^{1/3}$, f is certainly continuous on all of $\mathbb{R}^2$. Moreover,

$$\frac{\partial f}{\partial y} = (1/3)(ty)^{-2/3} \cdot \frac{\partial}{\partial y}[ty] = \frac{t}{3(ty)^{2/3}} = \frac{t^{1/3}}{3y^{2/3}}$$

which is continuous as long as $y \neq 0$, which it is not at the given point $(0,1)$.
(e) $\boxed{True:}$ For $y > 3$, $f(y) < 0$ and for $1 < y < 3$, $f(y) > 0$, so it is stable.

Problem 2: (30 points, 10 points each) **Short Answer** for the following problems. No justification is needed.

(a) $y_1 = e^{2t} + t$ and $y_2 = t$ are two known solutions to the linear differential equation $y'(t) + p(t)y(t) = f(t)$. Find the function $p(t)$.
(b) Given the initial value problem $y' = 2y, y(0) = 1$, use Euler's method to create an approximation to the solution at $t = 1$ using a step-size of $h = \frac{1}{2}$.
(c) The number of bacteria in a jar increases at a rate proportional to the population of bacteria in the jar. The number of bacteria in the jar triples in 4 hours. Then after 6 hours, the population is how many times of the original population?

Solution:

(a) $y = e^{2t} = y_1 - y_2$ is a solution to the homogeneous equation $y' + py = 0$. Plug in, we get $2e^{2t} + p(t)e^{2t} = 0 \Rightarrow \boxed{p(t) = -2}$.

(b) The solution is:

$$\begin{aligned} y_0 &= \; 1 \\ y_1 &= \; 1 + 2 * (1/2) = 2 \\ y_2 &= \; 2 + 2 * 2(1/2) = \boxed{4} \end{aligned}$$

(c) $\boxed{3^{3/2}}$. The population is $y(t) = y(0)e^{kt}$. So $y(4) = 3y(0) = y(0)e^{4k}$, therefore $4k = \ln(3)$, and so $y(6) = y(0)e^{6\ln(3)/4} = 3^{3/2}y(0)$.

Problem 3: (30 points) Given the differential equation $2y(y' + yt) = t(1 + y^2)$, answer the following questions:

(a) (5 points) Is the differential equation separable? If so, write it in separated form.
(b) (10 points) Are there any constant equilibrium solutions? If so, state the solution(s).
(c) (15 points) Solve the differential equation given the initial condition $y(0) = 2$.

Solution:

(a) Yes, the DE can be solved by separation of variables noticing that

$$2y(y' + yt) = t(1 + y^2) \Rightarrow \quad 2yy' + 2ty^2 = t + ty^2 \Rightarrow \quad 2yy' = t(1 - y^2) \Rightarrow$$
$$y' = t\left(\frac{1 - y^2}{2y}\right) = f(t)g(y).$$

(b) There are two constant equilibrium solutions. For $y = \pm 1$.

(c) By separation of variables, we need to solve

$$\int \frac{2ydy}{1 - y^2} = \int tdt + C.$$

The left hand side can be solved by using the substitution $u = 1 - y^2$ so that $du = -2ydy$

$$-\int \frac{du}{u} = -\ln|u| = -\ln|1 - y^2|.$$

The right hand side is solved by direct integration

$$\int tdt = t^2/2.$$

Then, the solution is

$$-\ln|1 - y^2| = t^2/2 + C, \Rightarrow \quad 1 - y^2 = Ke^{-t^2/2} \Rightarrow \quad y^2 = 1 - Ke^{-t^2/2}.$$

where $K = e^{-C}$.

Evaluating the initial condition,

$$y^2(0) = 4 = 1 - K \Rightarrow \quad K = -3.$$

Then the full solution is

$$y^2 = 1 + 3e^{-t^2/2}$$

Problem 4: (30 points) Use Variation of Parameters (Euler-Lagrange Method) to solve the following differential equation: $ty' + (2t + 1)y = 2$, where $t > 0$ and $y(1) = 1$.

(a) (12 points) Find all homogeneous solutions y_h.
(b) (12 points) Use Variation of Parameters (Euler-Lagrange) to find a particular solution y_p.
(c) (3 points) Find the general solution.
(d) (3 points) Find the solution that solves the IVP of the above equation with $y(1) = 1$.

Solution:

a First, normalize the equation to obtain $y' + \frac{2t+1}{t}y = \frac{2}{t}$ Then use separation of variables or the formula derived in class for $y' + p(t)y = f(t)$ to get: $y_h = Ce^{-\int p(t)dt} = Ce^{-\int(2+\frac{1}{t})dt} = \frac{C}{te^{2t}}$

b Let $y_p = \frac{v(t)}{te^{2t}}$, plug-in to the normalized differential equation $y' + \frac{2t+1}{t}y = \frac{2}{t}$, and solve for $v(t)$ giving $v(t) = \int 2e^{2t}dt = e^{2t}$. Thus, $y_p = \frac{v(t)}{te^{2t}} = \frac{1}{t}$

c $y = y_h + y_p = \frac{C}{te^{2t}} + \frac{1}{t} = \frac{Ce^{-2t}+1}{t}$

d Use $y(1) = 1$ in $y = \frac{Ce^{-2t}+1}{t}$ to solve for C, yielding $C = 0$, so the solution through (1,1) is $y = \frac{1}{t}$

Problem 5: (30 points, 15 points each) Salt water with a concentration of 1 lbs per gallon is pumped into a tank at a rate of 3 gallons per minute. Initially, the tank contains 50 gallons of fresh water. The well-stirred mixture flows out of the tank at the same rate of 3 gallons per minute.

(a) What is the initial value problem satisfied by the amount of salt in the tank $x(t)$?

(b) Solve the initial value problem in part (a) and find the amount of salt in the tank as a function of time.

Solution:

(a) Write down IVP

$dx/dt =$ concentration in$\times$ flow in - concentration out$\times$ flow out

$dx/dt =$ 1 lb/gal $\times$ 3 gals/min - $x/(50)$ lbs/gal $\times$ 3 gal/min.

$$\frac{dx}{dt} = \frac{3}{50}(50 - x), x(0) = 0 \tag{1}$$

(b) Solve for x(t)

- $x_p = 50$ is also the equilibrium solution.
- $x_h = c\exp^{-\frac{3}{50}t}$.
- $x_g = 50 + c\exp^{-\frac{3}{50}t}$, $x(0) = 0 \Rightarrow c = -50$

$\boxed{x(t) = 50(1 - \exp^{-\frac{3}{50}t})}$