Problem 1: (30 points)
Consider the following system of differential equations.

\[ \frac{dx}{dt} = x \left(y - x\right)^2 \]
\[ \frac{dy}{dt} = y - 1 \]

(a) Sketch the phase portrait of this system for \(-1 \leq x \leq 2\) and \(-1 \leq y \leq 2\), clearly labeling all nullclines, with arrows indicating the direction the solution will pass through them.

(b) Is \(\frac{dx}{dt}\) positive, negative, or zero at the point \((1.5, 0)\)?

(c) Identify/label all equilibrium points and classify their stability.

(d) What is the long term behavior of solutions with initial conditions having \(x < 0\) and \(y > 1\)?

Solution:

The equilibrium points are at \((0, 1)\) and \((1, 1)\).

(b) Positive.

(c) There are two equilibrium points. The equilibrium point at \((0, 1)\) is unstable and the equilibrium point at \((1, 1)\) is unstable.

(d) The solutions diverge towards positive infinity in the \(y\) direction and negative infinity in the \(x\) direction.
Problem 2: (30 points) Consider the linear system of equations \( Ax = b \), where
\[
A = \begin{bmatrix}
2 & 2 & 10 \\
0 & 1 & k+5 \\
1 & 1 & k+5
\end{bmatrix} \quad b = \begin{bmatrix}
2 \\
k - 1 \\
k + 1
\end{bmatrix}
\]

(a) Solve this system when \( k = 5 \) using Row Reduced Echelon Form (RREF)
(b) Calculate the determinant (for any constant \( k \))
(c) For which values of \( k \) does this system have (i) a unique solution, (ii) no solution, (iii) an infinite number of solutions.
(d) What is the column space of \( A \) (i.e. \( \text{col}(A) \)) if \(|A| \neq 0\).

Solution:

(a) 
\[
\begin{bmatrix}
2 & 2 & 10 \\
0 & 1 & 4 \\
1 & 1 & 6
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 5 \\
0 & 1 & 4 \\
1 & 1 & 6
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 5 \\
0 & 1 & 4 \\
0 & 0 & 5
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\Rightarrow x = \begin{bmatrix}
2 \\
-6 \\
1
\end{bmatrix}
\]

(b) 
\[
\begin{vmatrix}
2 & 2 & 10 \\
0 & 1 & k+5 \\
1 & 1 & k+5
\end{vmatrix} = 2 \begin{vmatrix}
1 & 1 & 5 \\
0 & 1 & k+5 \\
1 & 1 & k+5
\end{vmatrix} = 2 \begin{vmatrix}
1 & 1 & 5 \\
0 & 1 & 10 \\
0 & 0 & k
\end{vmatrix} = 2k
\]

By cofactors
\[
\begin{vmatrix}
2 & 2 & 10 \\
0 & 1 & k+5 \\
1 & 1 & k+5
\end{vmatrix} = 2 \begin{vmatrix}
1 & k+5 \\
1 & k+5 \\
1 & k+5
\end{vmatrix} + \begin{vmatrix}
2 & 10 \\
1 & k+5 \\
1 & k+5
\end{vmatrix} = 2k
\]

(c) (i) \( k \neq 0 \).
\[
\begin{bmatrix}
2 & 2 & 10 \\
0 & 1 & 5 \\
1 & 1 & 5
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 5 \\
0 & 1 & 5 \\
1 & 1 & 5
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 5 \\
0 & 1 & 5 \\
0 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 5 \\
0 & 0 & 0
\end{bmatrix}
\]

(ii) no value of \( k \) result in non-existence, (iii) \( k = 0 \).

(d) \( \text{col}(A) = R^3 \).
Problem 3: (30 points) In this problem, we consider $\mathbb{P}_3$, the vector space of polynomials with degree less or equal to 3. Justify all your answers.

(a) Is the following collection of polynomials a basis for $\mathbb{P}_3$?

\[ \left\{ 2t^3, (t^2 - 2t), (2t^2 + t), 3 \right\} \]

(b) What is the dimension of $\mathbb{P}_3$?

(c) We now consider a second collection of polynomials, indicated below. Is it a basis for $\mathbb{P}_3$?

\[ \left\{ (\sqrt{2}t^3 + \sqrt{3}t^2 - \sqrt{13}t), (\sqrt{3}t^3 + \sqrt{13}t^2 + \sqrt{2}), (t^2 - \sqrt{2}t + \sqrt{3}) \right\} \]

Solution:

(a) We first demonstrate that the collection of polynomials spans $\mathbb{P}_3$. Let $p(t) = b_0 + b_1t + b_2t^2 + b_3t^3$ be an arbitrary polynomial in $\mathbb{P}_3$, with $b_0, b_1, b_2, b_3 \in \mathbb{R}$. The existence of at least one solution $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that:

\[ 2\alpha_1 t^3 + \alpha_2 (t^2 - 2t) + \alpha_3 (2t^2 + t) + 3\alpha_4 = b_0 + b_1t + b_2t^2 + b_3t^3, \]

amounts to the existence of solutions for the system:

\[ \begin{align*}
2\alpha_1 &= b_3 \\
\alpha_2 + 2\alpha_3 &= b_2 \\
-2\alpha_2 + \alpha_3 &= b_1 \\
3\alpha_4 &= b_0
\end{align*} \]

which we can write matricially:

\[ \mathbb{A}\alpha = b, \quad \text{with} \quad \mathbb{A} = \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & -2 & 1 & 0 \\
0 & 0 & 0 & 3
\end{pmatrix} \]

Using the method of cofactors:

\[ \det (\mathbb{A}) = 2 \times \begin{vmatrix}
1 & 2 & 0 \\
-2 & 1 & 0 \\
0 & 0 & 3
\end{vmatrix} = 2 \times 3 \times \begin{vmatrix}
1 & 2 \\
-2 & 1
\end{vmatrix} = 2 \times 3 \times 5 = 30 \neq 0 \]

Hence $\mathbb{A}\alpha = b$ admits a solution, for any $b$: Our collection of polynomials span the entire space $\mathbb{P}_3$. Similarly, since $\det (\mathbb{A}) \neq 0$, the system $\mathbb{A}\alpha = 0$ admits the unique solution $\alpha = 0$: Our collection of polynomials is linearly independent. This is indeed a basis for $\mathbb{P}_3$.

(b) We exhibited a basis for $\mathbb{P}_3$ which contains 4 elements, hence:

\[ \dim (\mathbb{P}_3) = 4. \]

(c) This second collection of polynomials contains only 3 elements: a basis necessarily contains as many elements as the dimension of the space, which we just demonstrated to be $\dim (\mathbb{P}_3) = 4$. Therefore, this second collection of polynomials cannot be a basis for $\mathbb{P}_3$. 

Problem 4: (30 points)
Decide whether the given set $W$ is or is not a subspace of $V$. If $W$ is not a subspace, identify at least one requirement that is not satisfied. If $W$ is a subspace, prove it.

(a) $V = C^1(\mathbb{R})$ and $W = \{y(t) \mid y'(t) + \cos(y(t)) = 0\}$.
(b) $V = M_{2 \times 2}$ and $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\} \mid a, b, c, d \in \mathbb{R} \text{ and } a + d = 0 \}$.

Solution:

(a) $W$ is not a subspace of $V$.

Alternate solution 1: Let $y_1, y_2 \in W$. So $y_1'(t) + \cos(y_1(t)) = 0$ and $y_2'(t) + \cos(y_2(t)) = 0$. The sum $y_1 + y_2$ does not solve the differential equation: $(y_1 + y_2)'(t) + \cos(y_1(t) + y_2(t)) \neq 0$, thus $y_1 + y_2$ is not in $W$.

Alternate solution 2: Let $y_1 \in W$ and $c \in \mathbb{R}$. Then $y_1'(t) + \cos(y_1(t)) = 0$ but $cy_1$ does not solve the differential equation: $(cy_1)'(t) + \cos(cy_1(t)) \neq 0$ for any $c \in \mathbb{R}$, thus $cy_1$ is not in $W$.

(b) It is easy to check that $W \subset V$.

For any $A, B \in W$ and any $c, d \in \mathbb{R}$, we may assume that

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix},$$

where $a_i, b_i, c_i, d_i \in \mathbb{R}$ and $a_i + d_i = 0$ for any $i = 1, 2$.

Moreover,

$$cA + dB = \begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix} + \begin{bmatrix} da_2 & db_2 \\ dc_2 & dd_2 \end{bmatrix} = \begin{bmatrix} ca_1 + da_2 & cb_1 + db_2 \\ cc_1 + dc_2 & cd_1 + dd_2 \end{bmatrix} \in W \quad \text{Rubric : 8 pts}$$

since $ca_1 + da_2, cb_1 + db_2, cc_1 + dc_2, cd_1 + dd_2 \in \mathbb{R}$ and $(ca_1 + da_2) + (cd_1 + dd_2) = 0$. 
Problem 5: (30 points) Write TRUE or FALSE. No justification is required.

(a) For a square matrix $A$, $|A^T A| > 0$
(b) If $A$ and $B$ are $n \times n$ matrices with $\det A = 1$ and $\det B = 4$, then $\det(A + B) = 5$.
(c) The functions $\cos(x)$ and $\sin(x)$ are linearly independent elements of the vector space $V = C^1(\mathbb{R})$.
(d) If a solution of a linear system $A\vec{x} = \vec{b}$ exists, then it can be found as $\vec{x} = A^{-1}\vec{b}$, where $A^{-1}$ is the inverse of matrix $A$.
(e) Consider a linear system $A\vec{x} = \vec{0}$, where $A$ is a given $m \times n$ matrix and $\vec{x}$ is an $n$-vector. If $m > n$ and the system has a solution $\vec{x} \neq \vec{0}$, then $\text{RREF}(A)$ has at least $m - n$ zero rows.

Solution:

(a) FALSE. Since $|A^T| = |A|$, it’s true that $|A^T A| = |A^T| \cdot |A| = |A|^2$ but this is only greater than or equal to zero, as illustrated by $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
(b) FALSE. Use $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Then $\det(A + B) = 9$.
(c) TRUE: The Wronskian equals 1, so they are linearly independent.
(d) FALSE: true only if $A$ is a square matrix.
(e) TRUE: $m - n$ of the equations must be linear combinations of the others and the last $m - n$ rows of $\text{RREF}(A)$ are zeros.