1. (30 points) For the following two ODE’s, (i) find the **general** solution (ignoring initial conditions) and (ii) the **unique** solution satisfying the initial conditions provided.
   
   (a) (15 points)
   \[ y'' - 5y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = 5 \]

   (b) (15 points)
   \[ y'' + 2y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1 \]

2. (30 points) Consider the harmonic oscillator system:

   Assume that the mass of the block is \( m = 2 \), the spring constant \( k = 4 \), and that the equilibrium position of the block is \( x = 0 \). Both the damping constant \( b \) and the external force \( F(t) \) can vary. Let \( x(t) \) denote the position of the block at time \( t \). The initial conditions are \( x(0) = 1 \) and \( x'(0) = 1 \).

   (a) (2 points) Write down the differential equation for \( x(t) \), governing the motion of the block, using the supplied values of \( m \) and \( k \).

   (b) (8 points) Suppose the external force is zero and let \( b = 4 \). Write down the differential equation and find the solution \( x(t) \) using the supplied initial conditions. Describe the motion of the block.

   (c) (14 points) Suppose no damping is present \( (b = 0) \). An external force \( F(t) = \cos(t) \) is applied to the block. Write down the differential equation and derive the solution \( x(t) \) using the supplied initial conditions. Describe the motion of the block.

   (d) (6 points) Solving for \( x(t) \) is not necessary for this question.

   (I) (3 points) Suppose that \( F = 0 \) and we would like to design the damper so the block returns as fast as possible to the equilibrium position and ceases oscillation. What must the damping constant \( b \) be (or what range of values can it take) for this to be the case?

   (II) (3 points) Suppose no damping is present \( (b = 0) \) and a force \( F(t) = \cos(\sqrt{2}t) \) is applied. Describe the motion of the block in this case.

**CONTINUED**
3. (30 points)

(a) (6 points) True or False. If $y''_1 + 6y'_1 - 16y_1 = \frac{1}{t+1} e^{-t}$ and $y''_2 + 6y'_2 - 16y_2 = \cos(3t)$, then a general solution to the differential equation

$$y'' + 6y' - 16y = \frac{1}{t+1} e^{-t} + \cos(3t)$$

is $y = c_1 e^{2t} + c_2 e^{-8t} + y_1 + y_2$ for some constants $c_1$ and $c_2$.

For the next questions, you may use the method of undetermined coefficients without actually solving for the coefficients (however, you may wish to solve for the coefficients if you would like to check your answer).

(b) (8 points) Find the form of a particular solution to $y'' + 9y = \cos(3t + \pi/4)$.

(c) (8 points) Find the form of a particular solution to $y'' + 9y = t^2 e^{2t}$.

(d) Find the form of a particular solution to $y''' - 3y'' + 2y' - 4y = e^{2t}$

4. (30 points)

(a) (15 points) Find the Laplace transform $G(s)$ of

$$g(t) = e^t \sin(t)$$

(b) (15 points) Find $y(t)$ if its Laplace transform is

$$Y(s) = \frac{s - 1}{s^2 - 6s + 10}$$

5. (30 points) Consider the following IVP

$$y'' - 3y' + 2y = -2e^{-2t} \quad y(0) = 0, \quad y'(0) = 1$$

(a) (3 points) Given the Laplace transform $\mathcal{L}[y(t)] = Y(s)$, determine the form for $\mathcal{L}[y'(t)]$.

(b) (3 points) Given the Laplace transform $\mathcal{L}[y(t)] = Y(s)$, determine the form for $\mathcal{L}[y''(t)]$.

(c) (3 points) Compute the Laplace transform of $-2e^{-2t}$.

(d) (6 points) Using your answers from (a)-(c), find $Y(s)$ the Laplace transform to the solution of the IVP.

(e) (15 points) Find the solution to the IVP by taking the inverse Laplace transform of $Y(s)$.

Laplace transform table on next page
### Table 8.1.1 Short table of Laplace transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s) = \mathcal{L}[f(t)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) $1$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>(ii) $t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td>(iii) $e^{at}$</td>
<td>$\frac{1}{s-a}$</td>
</tr>
<tr>
<td>(iv) $t^ne^{at}$</td>
<td>$\frac{(s-a)^{n+1}}{n!}$</td>
</tr>
<tr>
<td>(v) $\sin bt$</td>
<td>$\frac{b}{s^2 + b^2}$</td>
</tr>
<tr>
<td>(vi) $\cos bt$</td>
<td>$\frac{s}{s^2 + b^2}$</td>
</tr>
<tr>
<td>(vii) $e^{at}\sin bt$</td>
<td>$\frac{(s-a)^2 + b^2}{s-a}$</td>
</tr>
<tr>
<td>(viii) $e^{at}\cos bt$</td>
<td>$\frac{(s-a)^2 + b^2}{b}$</td>
</tr>
<tr>
<td>(ix) $\sinh bt$</td>
<td>$\frac{b}{s^2 - b^2}$</td>
</tr>
<tr>
<td>(x) $\cosh bt$</td>
<td>$\frac{s}{s^2 - b^2}$</td>
</tr>
</tbody>
</table>