ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section, (4) your instructor’s name, and (5) a grading table for eight questions. Text books, class notes, and calculators are NOT permitted. A one-page two-sided crib sheet is allowed. In acknowledgment of attending the final exam, please sign and return your exam paper in your BLUEBOOK.

Problem 1 (30 points):
(i) (15 points) Solve the following ODE
\[ ty' = (t + 2)y + t^3 e^t \]

(ii) (15 points) Consider the nonlinear ODE
\[ y' = \frac{y^2 - t^2}{yt} \]
Using the \( v \)-substitution \( v = \frac{y}{t} \) find the first order ODE satisfied by \( v \). DO NOT solve the ODE.

Problem 2 (30 points): Alice is a graduate student in chemistry. In her lab she finds a tank that contains 10 gallons of chemical solution, which she labels as solution A. The chemical concentration of solution A is measured to be 2 pounds per gallon. She also finds plenty of chemical solution with chemical concentration 4 pounds per gallon, which she labels as solution B. To achieve the chemical concentration that she desires, she injects solution B into solution A at a flow rate of 3 gallons per hour, stirs the mixture uniformly, and drains the mixture out at a flow rate of 3 gallons per hour.

(i) (8 points) Set up the initial value problem for the mass of the chemical in the mixture as a function of time.

(ii) (12 points) Solve the initial value problem.

(iii) (6 points) Suppose that instead of draining the mixture at 3 gallons per hour, Alice evaporates the mixture at 3 gallons per hour. In this process, water escapes from the mixture while the chemical remains. In this case, what is the mass of the chemical in the mixture as a function of time?

(iv) (4 points) To reach a chemical concentration of 3 pounds per gallon, is it more efficient to evaporate the mixture or to drain the mixture? You do not need to justify your answer.

Problem 3 (30 points): Consider the second order non-homogeneous differential equation:
\[ (x - 1)y'' - xy' + y = (x - 1)^2. \]
Two solutions to the corresponding homogeneous problem are \( y_1(x) = x \) and \( y_2(x) = e^x \).

(a) (4 pts) Write the corresponding homogeneous equation. Verify that \( y_1(x) \) and \( y_2(x) \) solve the homogeneous equation.

(b) (4 pts) Show that \( y_1(x) \) and \( y_2(x) \) are linearly independent functions on \( x \in \mathbb{R} \).

(c)( 12 pts) Using the method of variation of parameters, find a particular solution to the non-homogeneous equation.

(d) (5 pts) Write the general solution to the non-homogeneous equation.

(e) (5 pts) Obtain the solution to the non-homogeneous equation with the initial condition \( y(0) = 1, y'(0) = 0 \).
Problem 4 (30 points)

(i) Write down the form of the particular solution for the following differential equations using the method of undetermined coefficients. Do not solve for the coefficients explicitly.
   (i) (5 points) \( y' - 3y = 3e^{2t}; \)
   (ii) (5 points) \( y'' - 2y' + y = te^t + t; \)
   (iii) (5 points) \( y''' + y' = 2 \sin t. \)

(ii) For certain linear inhomogeneous differential equations in applications, as time goes to infinity, the homogeneous solution decays to zero, and what remains is the particular solution known as the steady state. Determine the steady states of the following problems.
   (i) (5 points) Cooling: \( T' = -2(T - 50); \)
   (ii) (10 points) Forced oscillation: \( 2x'' + 4x' + 2x = 2 \cos t. \)

Problem 5 (35 points): Answer the following questions. Each question is worth 5 points. Only your final answers will be considered – no partial credit will be awarded for work.

(a) For the system of differential equations
   \[
   \frac{dx}{dt} = ye^{1+x^2+y^2}, \quad \frac{dy}{dt} = -xe^{1+x^2+y^2},
   \]
   (i) Find all equilibrium points.
   (ii) Find an implicit representation of the phase-plane trajectories.
   (iii) What do you think your answer to part (ii) says about the non-equilibrium solutions of the system?

(b) The system of differential equations
   \[
   \frac{dx}{dt} = -y - \frac{x(x^2 + y^2 - 2)}{\sqrt{x^2 + y^2}}, \quad \frac{dy}{dt} = x - \frac{y(x^2 + y^2 - 2)}{\sqrt{x^2 + y^2}}
   \]
   has one equilibrium point at \((0, 0)\) (it’s unstable) and an attracting limit cycle \( x^2 + y^2 = 2 \) (you do not have to verify any of this – take it as given).
   (i) What is the direction of motion of solution trajectories as they spiral into the limit cycle?
   (ii) Make a rough sketch of the phase plane that includes only the limit cycle and two solution trajectories – one that starts inside the limit cycle (but not at the origin) and one that starts outside the limit cycle. You do not have to find nullclines or a direction field.

(c) (i) Find a solution of the IVP \( y' = t\sqrt{1-y^2}, \quad y(0) = 1 \) other than equilibrium solution \( y(t) = 1. \) HINT: Recall that \( \frac{d}{du}[\arcsin(u)] = \frac{1}{\sqrt{1-u^2}}. \)
   (ii) Does your answer violate Picard’s Theorem? Answer YES or NO.
   (iii) Briefly explain your answer to part (ii).

(d) For \( m \) a positive constant, find the general solution of
   \[
   y''' - my'' + m^2 y' - m^3 y = 0
   \]
   HINT: factor by grouping.

(e) TRUE or FALSE? A “figure 8” (that is, a curve that looks like the infinity symbol \( \infty \)) could be a phase-plane solution trajectory of a system of differential equations
   \[
   \frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y)
   \]
   where \( f \) and \( g \) have continuous partial derivatives with respect to \( x \) and \( y. \)

(f) TRUE or FALSE? Every homogeneous linear differential equation with constant coefficients and purely imaginary roots of its characteristic equation must have periodic solutions.
TRUE OR FALSE? For every \( k > 0 \), the equilibrium solution \( x(t) = 0 \) of the nonlinear van der Pol equation
\[
\ddot{x} + k(x^2 - 1)\dot{x} + x = 0
\]
is unstable.

**Problem 6** (35 points): Answer “TRUE” or “FALSE”. You do NOT need to justify your answer. Only write “TRUE” if the statement is always true.

(i) Consider the two \( 5 \times 5 \) real valued matrices \( A \) and \( B \). If the vector \( v \) is an eigenvector of \( A \) with eigenvalue \( \lambda_A \), as well as an eigenvector of \( B \) with eigenvalue \( \lambda_B \). Then matrix \( C = A + B \) has an eigenvalue \( \lambda_A + \lambda_B \) and matrix \( D = AB \) has an eigenvalue \( \lambda_A \lambda_B \).

(ii) Consider the vectors \( v_1, v_2, v_3 \in \mathbb{R}^3 \) and the square matrix \( A \) formed by using the vectors as columns. If \( \det(A) = 0 \), then \( \dim(\text{span}\{v_1, v_2, v_3\}) < 3 \).

(iii) Consider a \( 4 \times 4 \) matrix \( A \). If \( \{v_1, v_2, v_3, v_4\} \) forms a basis of \( \text{Col}A \), then the matrix \( A \) is invertible.

(iv) Consider the two invertible \( n \times n \) matrices \( A \) and \( B \). If \( B^{-1}A^{-1}C = I \) and \( CB^{-1}A^{-1} = I \), then the \( n \times n \) matrix \( C \) is also invertible.

(v) Consider the three invertible \( n \times n \) matrices \( A, B \) and \( C \), then \( (AB)^T C = C^{-1} (A^{-1})^T (B^{-1})^T \).

(vi) The set of all solutions \( y(t) \) to the second order differential equation \( y'' + \sin(t)y' + t^3y = 0 \) forms a vector space.

(vii) Consider the \( n \times n \) real valued upper triangular matrix \( A \), if \( \det(A) \neq 0 \), then all eigenvalues are nonzero.

(viii) The set of functions \( \{1, x, 2x^2 - 1, 4x^3 - 3x\} \) is linearly independent.

**Problem 7** (points): Consider the following system of differential equations:
\[
\begin{pmatrix}
-1 & 6 & 2 \\
0 & 1 & 0 \\
2 & 6 & -1
\end{pmatrix}
\begin{pmatrix}
x(t)
\end{pmatrix}
\]

(i) Find three linearly independent solutions \( x_1(t), x_2(t), x_3(t) \).

(ii) Find the general solution \( x(t) \).

(iii) Find the particular solution for the initial condition \( x(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \).

**Problem 8** (points): Consider the following system of differential equations:
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -y + x - x^2
\end{align*}
\]

(i) Find all equilibrium points.

(ii) Determine the type of all equilibrium points.

(iii) Sketch the local behavior about the point \((1, 0)\).

(iv) Using the knowledge gained from (ii) and (iii). Sketch the phase plane.