Problem 1: (35 points) Let the matrix $A$ and column vector $\vec{b}$ be given by:

$$
A = \begin{bmatrix}
1 & -1 & -1 \\
-1 & 2 & 3 \\
1 & 1 & 4
\end{bmatrix}, \quad \vec{b} = \begin{pmatrix}
1 \\
0 \\
2
\end{pmatrix}
$$

(a) Compute $|A|$, the determinant of $A$.
(b) Is there a unique solution $\vec{x}$ to the system $A\vec{x} = \vec{b}$? Why or why not?
(c) Compute $A^{-1}$.
(d) Use $A^{-1}$ to solve the system $A\vec{x} = \vec{b}$.
(e) What is $(A^T)^{-1}$?
(f) What is $|A^{-1}|$?

Solution 1:

(a) 

$$
\begin{vmatrix}
1 & -1 & -1 \\
-1 & 2 & 3 \\
1 & 1 & 4
\end{vmatrix} = 1 \begin{vmatrix}
2 & 3 \\
1 & 4
\end{vmatrix} - (-1) \begin{vmatrix}
1 & 3 \\
1 & 4
\end{vmatrix} - 1 \begin{vmatrix}
1 & 2 \\
1 & 1
\end{vmatrix} = 5 - 7 + 3 = 1
$$

(b) Since the determinant of $A \neq 0$ the matrix is nonsingular (invertible), which means there will be a unique solution to the system for any righthand side.
(c) Compute $A^{-1}$ by using augmented matrix:

$$
A^{-1} = \begin{bmatrix}
5 & 3 & -1 \\
7 & 5 & -2 \\
-3 & -2 & 1
\end{bmatrix}
$$
\[ \vec{x} = A^{-1}\vec{b} \]
\[ \Rightarrow \vec{x} = \begin{bmatrix} 5 & 3 & -1 \\ 7 & 5 & -2 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} \]

(e)
\[ (A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 5 & 3 & -1 \\ 7 & 5 & -2 \\ -3 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 5 & 7 & -3 \\ 3 & 5 & -2 \\ -1 & -2 & 1 \end{bmatrix} \]

(f)
\[ |A^{-1}| = \frac{1}{|A|} = \frac{1}{(1)} = 1 \]

Problem 2: (30 points) Consider the following system of ODE's:
\[ \dot{x} = x(1 - x^2) \]
\[ \dot{y} = x - y \]

In answering the following questions, ensure that the graph is large enough to be read easily. In the x-y plane:
(a) Find and plot all equilibrium points.
(b) Find and plot the vertical nullclines. Indicate the direction of the vector field on the vertical nullclines.
(c) Find and plot the horizontal nullclines. Indicate the direction of the vector field on the horizontal nullclines.
(d) State the stability of all equilibrium points.
(e) Plot the solution with initial condition \((x_0, y_0) = (1, -1)\). Label this curve on the graph with the letter (e).
(f) Plot the solution with initial condition \((x_0, y_0) = (1/2, 1)\). Label this curve on the graph with the letter (f).

Solution 2:
The vector field is given as
\[ \mathbf{V} = [P(x, y), Q(x, y)] = (x(1 - x^2), x - y) \]
where \(P(x, y) = x(1 - x^2), Q(x, y) = x - y\).
(a) Equilibrium points occur for \(P(x, y) = x(1 - x^2) = 0\). \(Q(x, y) = x - y = 0\), thus
\[ x = 0, \quad \text{or} \quad x = \pm 1 \]
\[ y = x \]
Thus \((0, 0), (-1, -1), (1, 1)\).
(b) Vertical nullcline defined by \(P = x(1 - x^2) = 0\). Thus \(x = -1, x = 0, x = 1\) are the vertical nullclines and for
\[ x = -1, \quad \mathbf{V} = [0, -1 - y] \]
\[ x = 0, \quad \mathbf{V} = [0, -y] \]
x = 1, \[ V = [0, 1 - y] \]

(c) horizontal nullcline defined by \( Q = x - y = 0 \). Thus \( x = y \) are the horizontal nullclines and for \( x = y, \ V = [x(1 - x^2), 0] \)

(d) (0, 0) unstable, (−1, −1) stable, (1, 1) stable.

(e) see diagram

(f) see diagram

**Problem 3:** (30 points) Answer “TRUE” OR “FALSE”. You do NOT need to justify your answer. Only write TRUE if the statement is always true.

(a) Let \( A = \begin{bmatrix} 1 & 2 & 2 & -3 \\ -1 & 1 & -2 & 3 \\ 2 & 3 & 4 & -6 \end{bmatrix} \), then \( \dim(\text{Col} \ A) = 3 \)

(b) Consider the polynomial space \( \mathbb{P}_3 \) and \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{P}_3 \), then there exist \( \alpha, \beta \in \mathbb{R} \) such that \( \text{Span} \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \alpha \vec{v}_1 + \beta \vec{v}_2\} = \mathbb{P}_3 \)

(c) Let \( V \) be a vector space and \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \) be a basis for \( V \), then if \( \vec{w}_1, \vec{w}_2, \vec{w}_3 \in V \) and \( \text{Span} \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = V \), \( \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} \) is also a basis for \( V \)

(d) The set \( \{1 + t, 1 - t, 1 + t^3, 1 - t^3\} \) forms a basis for \( \mathbb{P}_3 \) on the interval \((-5, 5)\)

(e) Let \( A \) be a 5 × 5 matrix and \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} \) be a basis for \( \text{Col} \ A \), then \( A \) is singular

(f) Given matrices \( A, B, \) and \( C \) the following holds: \( |A B C B^{-1} A^{-1}| = |C| \)

**Solution 3:**

(a) False
(b) False
(c) True
(d) False
(e) True
(f) True
Problem 4: (20 points) Consider the following matrix:

\[
\begin{bmatrix}
0 & 0 & 4 & 5 \\
1 & k & 2 & -1 \\
0 & 0 & -4 & 0 \\
1 & -2 & 3k & 1
\end{bmatrix}
\]

(a) For what values of the constant \(k\) is the matrix \(C\) singular?

(b) For what values of the constant \(k\) is the matrix \(C\) invertible?

Solution 4:

(a) For the matrix \(C\) to be singular, \(\det(C) = 0\). Thus

\[
\begin{bmatrix}
0 & 0 & 4 & 5 \\
1 & k & 2 & -1 \\
0 & 0 & -4 & 0 \\
1 & -2 & 3k & 1
\end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 3k & 1 \\
1 & k & 2 & -1 \\
0 & 0 & -4 & 0 \\
0 & 0 & 4 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 3k & 1 \\
0 & k+2 & -1 & -2 \\
0 & 0 & -4 & 0 \\
0 & 0 & 0 & 5 \end{bmatrix}
\]

The final above matrix (denoted by \(B\)) is upper triangular and the determinant is simply the product of the diagonals such that \(\det(B) = (1)(k+2)(-4)(5) = -20(k+2)\). One row interchange was performed such that \(\det(C) = -\det(B) = 20(k+2)\). Thus, the matrix \(C\) is singular when \(k = -2\).

(b) The matrix is invertible when \(k \neq -2\).

Problem 5: (35 points)

(a) Consider each of the following sets. Decide whether each is a vector space or not. Explain your reasoning for each. (Hint: If the set is not a vector space, give at least one vector space property that fails.)

(i) The set of all polynomials of exactly degree 5.

(ii) The set of all invertible matrices.

(iii) The set of all continuous functions \(f\) defined on the interval \([0,1]\) such that \(f(0) = 1\).

(iv) The set of all integers.

(b) Consider the set of vectors given by

\[
\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

(i) Are \(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}\) linearly independent? Show your work.

(ii) What is the dimension of the span of \(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}\)?

(iii) Is the vector \(\vec{b} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}\) in the span of \(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}\)? Show your work.

Solution 5:

(a) None of the four sets is a vector space for the following reasons:

(i) The set of all polynomials of exactly degree 5 does not contain the zero vector, i.e. it does not contain the function that is identically equal to zero since the zero function is not of degree 5.
(ii) The set of all invertible matrices does not contain the zero matrix since the zero matrix is not invertible.

(iii) Suppose \( f \) and \( g \) are each elements of the set, i.e. they are both continuous on the interval \([0,1]\) and \(f(0) = 1\) and \(g(0) = 1\). Define \( h(x) = f(x) + g(x) \). \( h(0) = 1 + 1 = 2 \). Hence, \( h \) is not in the set so the set is not closed under vector addition.

(iv) The set of all integers is not a vector space since it is not closed under scalar multiplication. For example, \( c = 1/3 \) is a scalar but \( 2c = 2/3 \) is not an integer.

(b) For this part of the problem, we need to consider \( c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 \).

(i) If \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \) are linearly independent then the only solution to \( c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0} \) is \( c_1 = c_2 = c_3 = 0 \). To check this we compute:

\[
\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}
\]

We note that the only possible solutions is \( c_1 = c_2 = c_3 = 0 \) and the vectors are linearly independent.

(ii) The dimension of \( \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \) is 3.

(iii) If \( \vec{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} \) is in \( \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \) then there must exist constants \( c_1, c_2 \) and \( c_3 \) such that \( c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{b} \). We calculate:

\[
\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix}
\]

Thus, we have an inconsistent system and we conclude that \( \vec{b} \) is not in the span of \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \).