Problem 1: (35 points) Let the matrix $A$ and column vector $\vec{b}$ be given by:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

(a) Compute $|A|$, the determinant of $A$.

(b) Is there a unique solution $\vec{x}$ to the system $A\vec{x} = \vec{b}$? Why or why not?

(c) Compute $A^{-1}$.

(d) Use $A^{-1}$ to solve the system $A\vec{x} = \vec{b}$.

(e) What is $(A^T)^{-1}$?

(f) What is $|A^{-1}|$?

Problem 2: (30 points) Consider the following system of ODE’s:

$$\dot{x} = x(1-x^2)$$
$$\dot{y} = x - y$$

In answering the following questions, ensure that the graph is large enough to be read easily. In the $x$-$y$ plane:

(a) Find and plot all equilibrium points.

(b) Find and plot the vertical nullclines. Indicate the direction of the vector field on the vertical nullclines.

(c) Find and plot the horizontal nullclines. Indicate the direction of the vector field on the horizontal nullclines.

(d) State the stability of all equilibrium points.

(e) Plot the solution with initial condition $(x_0, y_0) = (1, -1)$. Label this curve on the graph with the letter (e).

(f) Plot the solution with initial condition $(x_0, y_0) = (1/2, 1)$. Label this curve on the graph with the letter (f).

Problem 3: (30 points) Answer “TRUE” OR “FALSE”. You do NOT need to justify your answer. Only write TRUE if the statement is always true.

(a) Let $A = \begin{bmatrix} 1 & 2 & 2 & -3 \\ -1 & 1 & -2 & 3 \\ 2 & 3 & 4 & -6 \end{bmatrix}$, then $\dim(\text{Col } A) = 3$

(b) Consider the polynomial space $\mathbb{P}_3$ and $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{P}_3$, then there exist $\alpha, \beta \in \mathbb{R}$ such that $\text{Span } \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \alpha \vec{v}_1 + \beta \vec{v}_2\} = \mathbb{P}_3$

(c) Let $V$ be a vector space and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for $V$, then if $\vec{w}_1, \vec{w}_2, \vec{w}_3 \in V$ and $\text{Span } \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = V$, $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is also a basis for $V$.
(d) The set \( \{1 + t, 1 - t, 1 + t^3, 1 - t^3\} \) forms a basis for \( \mathbb{P}_3 \) on the interval \((-5, 5)\)

(e) Let \( \mathbf{A} \) be a \( 5 \times 5 \) matrix and \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} \) be a basis for \( \text{Col} \; \mathbf{A} \), then \( \mathbf{A} \) is singular

(f) Given matrices \( \mathbf{A}, \mathbf{B}, \) and \( \mathbf{C} \) the following holds: \( |\mathbf{A}\mathbf{B}\mathbf{C}^{-1}\mathbf{A}^{-1}| = |\mathbf{C}| \)

**Problem 4:** (20 points) Consider the following matrix:

\[
\mathbf{C} = \begin{bmatrix}
0 & 0 & 4 & 5 \\
1 & k & 2 & -1 \\
0 & 0 & -4 & 0 \\
1 & -2 & 3k & 1
\end{bmatrix}
\]

(a) For what values of the constant \( k \) is the matrix \( \mathbf{C} \) singular?

(b) For what values of the constant \( k \) is the matrix \( \mathbf{C} \) invertible?

**Problem 5:** (35 points)

(a) Consider each of the following sets. Decide whether each is a vector space or not. Explain your reasoning for each. (Hint: If the set is not a vector space, give at least one vector space property that fails.)

(i) The set of all polynomials of exactly degree 5.

(ii) The set of all invertible matrices.

(iii) The set of all continuous functions \( f \) defined on the interval \([0, 1]\) such that \( f(0) = 1 \).

(iv) The set of all integers.

(b) Consider the set of vectors given by \( \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \), \( \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \), \( \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \)

(i) Are \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \) linearly independent? Show your work.

(ii) What is the dimension of the span of \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \)?

(iii) Is the vector \( \vec{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} \) in the span of \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \)? Show your work.