APPM 2360: FINAL EXAM
Dec. 16, 2017

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) "Final"/instructors name, (3) your **LECTURE** section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A letter sized crib sheet is allowed.

Solution: APPM 2360 **FINAL EXAM** **Fall 2017**

Problem 1: (36 points) **True/False** (answer True if it is always true otherwise answer False. No justification is needed.)

(a) For invertible matrices A, B, and C , if $AB = CA$ then $|B| = |C|$.
(b) The space of all solutions to the differential equation $y'' - \cos t^2 y' + y - 5t = 0$ forms a vector space (usual addition and scalar multiplication are assumed).
(c) A is a 3×4 matrix, when the system of linear equations $A\bar{x} = \bar{b}$ is solvable, then we must have infinitely many solutions. Here $\bar{x}$ is a 4×1 column vector and $\bar{b}$ is a 3×1 column vector.
(d) The critical (or equilibrium) point $x_0 = 1$ of the differential equation $x'(t) = x(1 - x)$ is unstable.
(e) The dimension of the vector space $\mathbb{U}_{4\times4} = \{4\times4 \text{ upper triangular matrices with real entries}\}$ is 10.
(f) If $f(y, t)$ is continuous everywhere, Picard's theorem guarantees that the differential equation $\frac{dy}{dt} = f(y, t)$ has a unique solution for any initial condition $y(t_0) = y_0$.

Solution:

(a) True,

$$|AB| = |CA| \rightarrow |A||B| = |C||A| \rightarrow |B| = |C|$$

(b) False, This equation is nonhomogeneous and so the solution space does not form a vector space, e.g., the zero function is not in the space.
(c) True, $rank(A) \leq 3 < 4 =$ number of variables.
(d) False, $f(x) = x - x^2$ and $f'(1) = -1 < 0$, it is stable.
(e) True, there are 10 entries above the diagonal that can be chosen independently. A basis is the set of 10 matrices having only one entry on or above the diagonal equal to 1, and the rest of the entries 0.
(f) False, this is guaranteed by Picard's theorem only if $\frac{\partial f}{\partial y}$ is continuous everywhere.

Problem 2: (30 points) **Short Answer** for the following problems. Box your answer. No work for this question will be graded.

(a) For the IVP $y' = 2t/(1 + 2y), y(0) = 0$ approximate the solution at $t = 1$ using Euler's method with a step size $h = 0.5$.
(b) The vector space $\mathbb{M}_{32}$ consists of all 3×2 matrices. Consider the subset $\mathbb{W}$ which contains matrices with zeros in the second row i.e. $a_{21} = a_{22} = 0$. Is $\mathbb{W}$ a vector subspace of $\mathbb{M}_{32}$? If so, prove it. If not, give one counterexample.
(c) The reduced row-echelon form of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

is

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Compute the coefficients to express the last column of A through the basis vectors of the column space of A.

Solution:

(a) Euler's method is given by: $t_{n+1} = t_n + h$, $y_{n+1} = y_n + hf(t_n, y_n)$ for $n = 0, 1, 2, \ldots$ Then notice

$$\begin{aligned} t_0 &= 0, & y_0 &= y(0) = 0, \\ t_1 &= t_0 + h & y_1 &= y_0 + hf(t_0, y_0) \\ &= 0 + 0.5 & &= 0 + 0.5[2 * 0/(1 + 2 * 0)] \\ &= 0.5, & &= 0, \\ t_2 &= t_1 + h & y_2 &= y_1 + hf(t_1, y_1) \\ &= 0.5 + 0.5 & &= 0 + 0.5[2 * 0.5/(1 + 2 * 0)] \\ &= 1, & &= 0.5, \\ t_3 &= t_2 + h & y_3 &= y_2 + hf(t_2, y_2) \\ &= 1 + 0.5 & &= 0.5 + 0.5[2 * 1/(1 + 2 * 0.5)] \\ &= 1.5. & &= 1. \end{aligned}$$

(b) Any nonempty subset of a vector space is a subspace if it is closed under addition and scalar multiplication. The subset $\mathbb{W}$ is a subspace of $\mathbb{M}_{32}$. Consider the linear combination $\alpha A + \beta B$ where $\alpha, \beta \in \mathbb{R}$ and $A, B \in \mathbb{W}$ i.e. $a_{21} = a_{22} = b_{21} = b_{22} = 0$. Then notice

$$\alpha A + \beta B = \alpha \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \\ a_{31} & a_{32} \end{bmatrix} + \beta \begin{bmatrix} b_{11} & b_{12} \\ 0 & 0 \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} + \beta b_{11} & \alpha a_{12} + \beta b_{12} \\ 0 & 0 \\ \alpha a_{31} + \beta b_{31} & \alpha a_{32} + \beta b_{32} \end{bmatrix}.$$

Since the resulting matrix has zeros along the second row we see that $\alpha A + \beta B \in \mathbb{W}$ and conclude $\mathbb{W}$ is a subspace.

(c)

We can choose basis vectors as

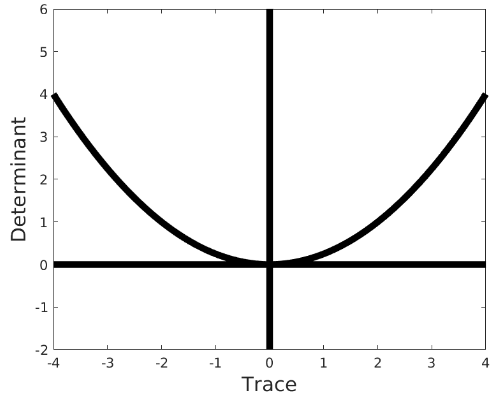
$$\begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix},$$

since RREF has non-zero pivots in these columns. Thus, we look for coefficients a and b such that

$$a \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix} + b \begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$$

and find that $a = -2$ and $b = 3$.

Problem 3: (30 points) Below is the trace/determinant plane for $\bar{x}' = A\bar{x}$ where A is a 2×2 matrix of real numbers. For your convenience, the figure also contains curves that define the boundaries of behavior for different solutions.



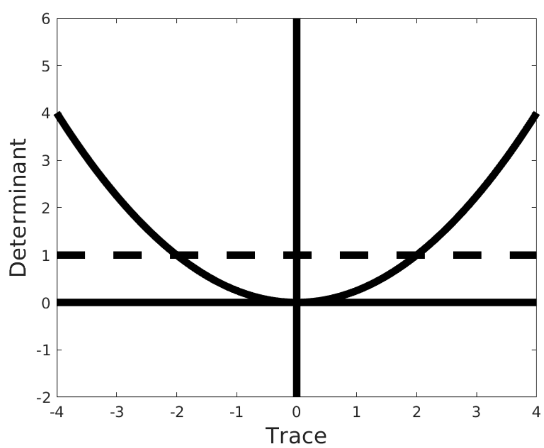
Let A be the matrix

$$A = \begin{bmatrix} a-1 & 1 \\ a-2 & 1 \end{bmatrix}.$$

where the (1,1) and (2,1) elements of A depend on the value of a parameter a. If a can be any real number, describe the different classifications that the steady state at $\bar{x}_* = (0, 0)$ can have.
Solution:

Attracting sink, repelling source, degenerate (attracting and repelling), spirals (attracting and repelling), and center

To solve this problem, we'll need to compute both the Trace and determinant of the matrix A. The $\text{Tr}(A) = a$, while $|A| = (a-1) - (a-2) = 1$. Therefore, we need to graph the parametric plot $(a, 1)$ as a is varied



As is illustrated in the graph, the parametric curve is the dotted, horizontal line, intersecting the vertical axis at 1. Thus the node at (0,0) can be classified (moving from left to right along the dotted line) as attracting sink, attracting degenerate, attracting spiral, center, repelling spiral, repelling degenerate, and repelling source.

Problem 4: (35 points) Consider the linear system $A\bar{x} = \bar{b}$ where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \ \bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ \bar{b} = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}.$$

(a) Calculate the determinant of A.
(b) Is there a unique solution of $A\bar{x} = \bar{b}$?
(c) Put the augmented matrix in RREF.
(d) What is the solution, if it exists.
(e) Geometrically, in $\mathbb{R}^3$, does the solution to $A\bar{x} = \bar{b}$ correspond to a point, a line, or a plane?

Solution:
(a) Expanding along the second row gives

$$|A| = -(-2)\begin{vmatrix} -1 & 1 \\ -3 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = 2(-4+3) + (4-2) = 0\,.$$

(b) Since $|A| = 0$ there is no unique solution.
(c) Notice that

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ -2 & 1 & 0 & -2 \\ 2 & -3 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & -1 & 2 & 2 \\ 0 & -1 & 2 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & 2 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

(d) The corresponding equations are: $x - z = 0, y - 2z = -2$, where z is a free parameter. The infinite number of solutions are given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ -2+2\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \alpha \in \mathbb{R}.$$

(e) The equations $x - z = 0$ and $y - 2z = -2$ correspond to non-parallel planes that intersect in $\mathbb{R}^3$. Hence the solution corresponds to the line of intersection of these two planes that passes through the point (0,-2,0) and is parallel to the vector $\langle 1, 2, 1 \rangle$.

Problem 5: (40 points)
Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues of A.
(b) Find the eigenvectors of A.

Solution:

(a) The characteristic equation is

$$|A - \lambda I| = \left|\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & -1 \\ 0 & 1 & 2-\lambda \end{bmatrix}\right| = (1-\lambda)[(2-\lambda)^2 + 1] = 0.$$

The roots are $\lambda = 1$ and $\lambda = 2 \pm i$. Therefore the eigenvalues are $\lambda_1 = 1$, $\lambda_{2,3} = 2 \pm i$.

(b) For $\lambda_1 = 1$, we solve $(A - I)\vec{u}_1 = \vec{0}$. We write that in augmented matrix form and do row operations as follows

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right].$$

If we say $\vec{u}_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the system is $2c = 0$, $b - c = 0$, and $b + c = 0$. These equations give us $b = c = 0$, leaving a a free variable. Choosing $a = 1$, one eigenvector of A corresponding to $\lambda_1 = 1$ is $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

For $\lambda_2 = 2 + i$, we solve the system $(A - (2 + i)I)\vec{u}_2 = \vec{0}$:

$$\left[\begin{array}{ccc|c} -1-i & 1 & 1 & 0 \\ 0 & -i & -1 & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \xrightarrow[iR_2 \to R_2]{R_3 - iR_2} \left[\begin{array}{ccc|c} -1-i & 1 & 1 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Now we say $\vec{u}_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and use backsubstitution. The second equation gives $b - ic = 0$, so $b = ic$. The first equation gives $-(1 + i)a + b + c = 0$, so $(1 + i)a = c + ic = (1 + i)c$, so $a = c$. So $\vec{u}_2 = \begin{bmatrix} c \\ ic \\ c \end{bmatrix}$. Choosing $c = 1$, one eigenvector of A corresponding to $\lambda_2 = 2 + i$ is $\vec{u}_2 = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}$. Since the matrix is real, the remaining eigenvector is the complex conjugate of the one we just found,

$$\vec{u}_3 = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$$

.

Problem 6: (40 points) A mass of 1 kg is attached to a spring with constant $k = 25$ N/m.

(a) Assuming there is no damping, i.e., $b = 0$ and assuming the system is forced with a forcing term of the form $f(t) = 20\cos(\omega_{\text{f}}\, t)$ (measured in Newtons). Find a value of ω_f that guarantees that the amplitude of the resulting oscillations grows without limit.

(b) Assuming the system is forced with a forcing term $f(t) = 102\cos(t)$ (measured in Newtons). We also assume that the corresponding damping constants is $b = 6$.
   (i) Write down the second order differential equation for this mass-spring motion.
   (ii) Verify that $x_p(t) = 4\cos(t) + \sin(t)$ is a particular solution to the above second order differential equation.
   (iii) Find the general solution.
   (iv) Find the frequency of the steady-state for this motion.

Solution:

(a) $\omega_f = \omega_0 = 5$.

(b) $m = 1$, $b = 6$, $k = 25$, $f = 102\cos(t)$, $x(0) = x'(0) = 0$:
   (i) $x''(t) + 6x'(t) + 25x(t) = 102\cos(t)$; $x'(0) = x(0) = 0$.
   (ii) Direct computing: $x_p'(t) = -4\sin(t) + \cos(t)$, $x_p''(t) = -4\cos(t) - \sin(t)$, and plug in.
   (iii) $r^2 + 6r + 25 = 0 \Rightarrow r = -3 \pm 4i$: $x = \exp^{-3t}(A\cos(4t) + B\sin(4t)) + 4\cos(t) + \sin(t)$.
   (iv) $\omega = 1$.

Problem 7: (39 points)
Use Laplace transform to solve the initial value problem

$$y'' + 4y = \cos(3t),$$

where $y(0) = 1$ and $y'(0) = 0$.

(a) Derive an algebraic equation for the Laplace transform of y, $Y(s) = \mathcal{L}(y)(s)$.
(b) Solve the algebraic equation from a).
(c) Evaluate the inverse Laplace transform and obtain the solution of the initial value problem.

Solution:

(a) We have

$$\left(s^2\mathcal{L}(y) - s\right) + 4\mathcal{L}(y) = \frac{s}{s^2+9}$$

(b) We obtain

$$\mathcal{L}(y)(s) = \frac{s}{(s^2+9)(s^2+4)} + \frac{s}{s^2+4}$$

(c) Computing

$$\frac{s}{(s^2+9)(s^2+4)} = \frac{1}{5}\left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right)$$

so that

$$\mathcal{L}(y)(s) = \frac{6}{5}\frac{s}{s^2+4} - \frac{1}{5}\frac{s}{s^2+9},$$

we obtain

$$y(t) = \frac{6}{5}\cos(2t) - \frac{1}{5}\cos(3t).$$

It easy to check that the resulting solution satisfies both, the equation and the initial conditions.

Table 8.1.1 Short table of Laplace transforms

	$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	
(i)	1	$\frac{1}{s}$	$s > 0$
(ii)	t^n	$\frac{n!}{s^{n+1}}$	$s > 0$, n a positive integer
(iii)	e^{at}	$\frac{1}{s-a}$	$s > a$
(iv)	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$, n a positive integer
(v)	$\sin bt$	$\frac{b}{s^2+b^2}$	$s > 0$
(vi)	$\cos bt$	$\frac{s}{s^2+b^2}$	$s > 0$
(vii)	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
(viii)	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
(ix)	$\sinh bt$	$\frac{b}{s^2-b^2}$	$s > \|b\|$
(x)	$\cosh bt$	$\frac{s}{s^2-b^2}$	$s > \|b\|$