Problem 1: (36 points) True/False (answer True if it is always true otherwise answer False. No justification is needed.)

(a) For invertible matrices $A$, $B$, and $C$, if $AB = CA$ then $|B| = |C|$.
(b) The space of all solutions to the differential equation $y'' - \cos t^2 y' + y - 5t = 0$ forms a vector space (usual addition and scalar multiplication are assumed).
(c) $A$ is a $3 \times 4$ matrix, when the system of linear equations $A\bar{x} = \bar{b}$ is solvable, then we must have infinitely many solutions. Here $\bar{x}$ is a $4 \times 1$ column vector and $\bar{b}$ is a $3 \times 1$ column vector.
(d) The critical (or equilibrium) point $x_0 = 1$ of the differential equation $x'(t) = x(1 - x)$ is unstable.
(e) The dimension of the vector space $U_{4\times4} = \{4\times4 \text{ upper triangular matrices with real entries}\}$ is 10.
(f) If $f(y, t)$ is continuous everywhere, Picard’s theorem guarantees that the differential equation $\frac{dy}{dt} = f(y, t)$ has a unique solution for any initial condition $y(t_0) = y_0$.

Problem 2: (30 points) Short Answer for the following problems. [Box your answer.] No work for this question will be graded.

(a) For the IVP $y' = 2t/(1 + 2y), y(0) = 0$ approximate the solution at $t = 1$ using Euler’s method with a step size $h = 0.5$.
(b) The vector space $\mathbb{M}_{32}$ consists of all $3 \times 2$ matrices. Consider the subset $\mathbb{W}$ which contains matrices with zeros in the second row i.e. $a_{21} = a_{22} = 0$. Is $\mathbb{W}$ a vector subspace of $\mathbb{M}_{32}$? If so, prove it. If not, give one counterexample.
(c) The reduced row-echelon form of the matrix

$$A = \begin{pmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12
\end{pmatrix}$$

is

$$\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$ Compute the coefficients to express the last column of $A$ through the basis vectors of the column space of $A$.  


**Problem 3:** (30 points) Below is the trace/determinant plane for $\ddot{x} = A\dot{x}$ where $A$ is a $2 \times 2$ matrix of real numbers. For your convenience, the figure also contains curves that define the boundaries of behavior for different solutions.

![Trace/Determinant Plane](image)

Let $A$ be the matrix

$$A = \begin{bmatrix} a - 1 & 1 \\ a - 2 & 1 \end{bmatrix}.$$

where the (1,1) and (2,1) elements of $A$ depend on the value of a parameter $a$. If $a$ can be any real number, describe the different classifications that the steady state at $\bar{x} = (0,0)$ can have.

**Problem 4:** (35 points) Consider the linear system $A\bar{x} = \bar{b}$ where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}.$$

(a) Calculate the determinant of $A$.
(b) Is there a unique solution of $A\bar{x} = \bar{b}$?
(c) Put the augmented matrix in RREF.
(d) What is the solution, if it exists.
(e) Geometrically, in $\mathbb{R}^3$, does the solution to $A\bar{x} = \bar{b}$ correspond to a point, a line, or a plane?

**Problem 5:** (40 points)
Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues of $A$.
(b) Find the eigenvectors of $A$.

**Problem 6:** (40 points) A mass of 1 kg is attached to a spring with constant $k = 25$ N/m.

(a) Assuming there is no damping, i.e., $b = 0$ and assuming the system is forced with a forcing term of the form $f(t) = 20\cos(\omega_f t)$ (measured in Newtons). Find a value of $\omega_f$ that guarantees that the amplitude of the resulting oscillations grows without limit.
(b) Assuming the system is forced with a forcing term $f(t) = 102\cos(t)$ (measured in Newtons). We also assume that the corresponding damping constants is $b = 6$.
   (i) Write down the second order differential equation for this mass-spring motion.
   (ii) Verify that $x_p(t) = 4\cos(t) + \sin(t)$ is a particular solution to the above second order differential equation.
   (iii) Find the general solution.
   (iv) Find the frequency of the steady-state for this motion.
Problem 7: (39 points)
Use Laplace transform to solve the initial value problem
\[ y'' + 4y = \cos(3t), \]
where \( y(0) = 1 \) and \( y'(0) = 0 \).
(a) Derive an algebraic equation for the Laplace transform of \( y \), \( Y(s) = \mathcal{L}(y)(s) \).
(b) Solve the algebraic equation from a).
(c) Evaluate the inverse Laplace transform and obtain the solution of the initial value problem.

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( F(s) = \mathcal{L}[f(t)] )</th>
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</thead>
<tbody>
<tr>
<td>(i) 1</td>
<td>( \frac{1}{s} ) s &gt; 0</td>
</tr>
<tr>
<td>(ii) ( t^n )</td>
<td>( \frac{n!}{s^{n+1}} ) s &gt; 0, ( n ) a positive integer</td>
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<tr>
<td>(iii) ( e^{at} )</td>
<td>( \frac{1}{s-a} ) s &gt; a</td>
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<tr>
<td>(iv) ( t^n e^{at} )</td>
<td>( \frac{n!}{(s-a)^{n+1}} ) s &gt; a, ( n ) a positive integer</td>
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<tr>
<td>(v) ( \sin bt )</td>
<td>( \frac{b}{s^2 + b^2} ) s &gt; 0</td>
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<tr>
<td>(vi) ( \cos bt )</td>
<td>( \frac{s}{s^2 + b^2} ) s &gt; 0</td>
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<tr>
<td>(vii) ( e^{at} \sin bt )</td>
<td>( \frac{(s-a)^2 + b^2}{s-a} ) s &gt; a</td>
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<tr>
<td>(viii) ( e^{at} \cos bt )</td>
<td>( \frac{(s-a)^2 + b^2}{b} ) s &gt; a</td>
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<tr>
<td>(ix) ( \sinh bt )</td>
<td>( \frac{s^2 - b^2}{s} ) s &gt; (</td>
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<tr>
<td>(x) ( \cosh bt )</td>
<td>( \frac{s^2 - b^2}{s} ) s &gt; (</td>
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