APPM 2360: FINAL EXAM

Dec. 16, 2017

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) "Final"/instructors name, (3) your **LECTURE** section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A letter sized crib sheet is allowed.

Problem 1: (36 points) **True/False** (answer True if it is always true otherwise answer False. No justification is needed.)

(a) For invertible matrices A, B, and C , if $AB = CA$ then $|B| = |C|$.

(b) The space of all solutions to the differential equation $y'' - \cos t^2 y' + y - 5t = 0$ forms a vector space (usual addition and scalar multiplication are assumed).

(c) A is a 3×4 matrix, when the system of linear equations $A\bar{x} = \bar{b}$ is solvable, then we must have infinitely many solutions. Here $\bar{x}$ is a 4×1 column vector and $\bar{b}$ is a 3×1 column vector.

(d) The critical (or equilibrium) point $x_0 = 1$ of the differential equation $x'(t) = x(1 - x)$ is unstable.

(e) The dimension of the vector space $\mathbb{U}_{4\times 4} = \{4\times 4 \text{ upper triangular matrices with real entries}\}$ is 10.

(f) If $f(y, t)$ is continuous everywhere, Picard's theorem guarantees that the differential equation $\frac{dy}{dt} = f(y, t)$ has a unique solution for any initial condition $y(t_0) = y_0$.

Problem 2: (30 points) **Short Answer** for the following problems. Box your answer. No work for this question will be graded.

(a) For the IVP $y' = 2t/(1 + 2y), y(0) = 0$ approximate the solution at $t = 1$ using Euler's method with a step size $h = 0.5$.

(b) The vector space $\mathbb{M}_{32}$ consists of all 3×2 matrices. Consider the subset $\mathbb{W}$ which contains matrices with zeros in the second row i.e. $a_{21} = a_{22} = 0$. Is $\mathbb{W}$ a vector subspace of $\mathbb{M}_{32}$? If so, prove it. If not, give one counterexample.

(c) The reduced row-echelon form of the matrix

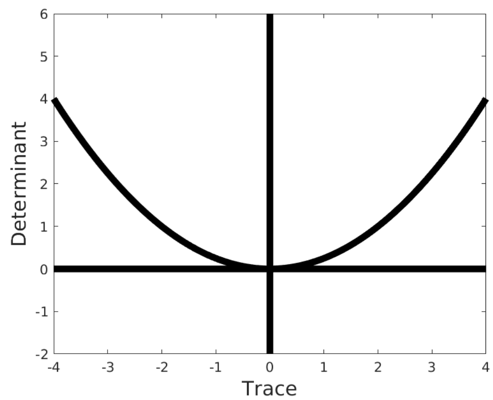
$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

is

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Compute the coefficients to express the last column of A through the basis vectors of the column space of A.

Problem 3: (30 points) Below is the trace/determinant plane for $\bar{x}' = A\bar{x}$ where A is a 2×2 matrix of real numbers. For your convenience, the figure also contains curves that define the boundaries of behavior for different solutions.



Let A be the matrix

$$A = \begin{bmatrix} a-1 & 1 \\ a-2 & 1 \end{bmatrix}.$$

where the (1,1) and (2,1) elements of A depend on the value of a parameter a. If a can be any real number, describe the different classifications that the steady state at $\bar{x}_* = (0,0)$ can have.

Problem 4: (35 points) Consider the linear system $A\bar{x} = \bar{b}$ where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \; \bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; \bar{b} = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}.$$

(a) Calculate the determinant of A.
(b) Is there a unique solution of $A\bar{x} = \bar{b}$?
(c) Put the augmented matrix in RREF.
(d) What is the solution, if it exists.
(e) Geometrically, in $\mathbb{R}^3$, does the solution to $A\bar{x} = \bar{b}$ correspond to a point, a line, or a plane?

Problem 5: (40 points)
Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues of A.
(b) Find the eigenvectors of A.

Problem 6: (40 points) A mass of 1 kg is attached to a spring with constant $k = 25$ N/m.

(a) Assuming there is no damping, i.e., $b = 0$ and assuming the system is forced with a forcing term of the form $f(t) = 20\cos(\omega_f\, t)$ (measured in Newtons). Find a value of ω_f that guarantees that the amplitude of the resulting oscillations grows without limit.
(b) Assuming the system is forced with a forcing term $f(t) = 102\cos(t)$ (measured in Newtons). We also assume that the corresponding damping constants is $b = 6$.
  (i) Write down the second order differential equation for this mass-spring motion.
  (ii) Verify that $x_p(t) = 4\cos(t) + \sin(t)$ is a particular solution to the above second order differential equation.
  (iii) Find the general solution.
  (iv) Find the frequency of the steady-state for this motion.

Problem 7: (39 points)
Use Laplace transform to solve the initial value problem

$$y'' + 4y = \cos(3t),$$

where $y(0) = 1$ and $y'(0) = 0$.

(a) Derive an algebraic equation for the Laplace transform of y, $Y(s) = \mathcal{L}(y)(s)$.
(b) Solve the algebraic equation from a).
(c) Evaluate the inverse Laplace transform and obtain the solution of the initial value problem.

Table 8.1.1 Short table of Laplace transforms

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	
(i) 1	$\frac{1}{s}$	$s > 0$
(ii) t^n	$\frac{n!}{s^{n+1}}$	$s > 0$, n a positive integer
(iii) e^{at}	$\frac{1}{s-a}$	$s > a$
(iv) $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$, n a positive integer
(v) $\sin bt$	$\frac{b}{s^2+b^2}$	$s > 0$
(vi) $\cos bt$	$\frac{s}{s^2+b^2}$	$s > 0$
(vii) $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
(viii) $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
(ix) $\sinh bt$	$\frac{b}{s^2-b^2}$	$s > \lvert b\rvert$
(x) $\cosh bt$	$\frac{s}{s^2-b^2}$	$s > \lvert b\rvert$