APPM 2360: Midterm exam 1
September 27, 2017

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your **LECTURE** section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one page letter sized crib sheet is allowed.

Solution: APPM 2360 **Exam1** **Fall 2017**

Problem 1: (30 points) **True/False** (answer True if it is always true otherwise answer False. No justification is needed.)

(a) (7pts) The equilibrium solution $y(t) \equiv 100$ of the logistic problem $y'(t) = 1/10(1 - \frac{y}{100})y$ is unstable.

(b) (7pts) The differential equation $y''(t) + t^2 y'(t) + e^t y(t) = t^3$ is nonlinear.

(c) (8pts) If $y_1(t) \neq y_2(t)$ are two solutions of $y'(t) + e^t y(t) = sin^2 t$ then $y(t) = C(y_1(t) - y_2(t))$ with $C \in \mathbb{R}$ is the general solution of $y'(t) + e^t y(t) = 0$.

(d) (8pts) If $y(t) \equiv y_0$ is an equilibrium solution to $y'(t) = f(y)$, then $y_n = y_0$ when we use Euler's method to solve the initial value problem $y'(t) = f(y)$, $y(0) = y_0$.

Solution:

(a) False, $f'(100) = -0.1 < 0$; or near $y = 100$, $f(y) < 0$ for $y > 100$ and $f(y) > 0$ for $y < 100$, it is stable.

(b) False.

(c) TRUE. $y_1(t) - y_2(t) \neq 0$ solves the homogenenous equation since, $y_1'(t) + p(t)y_1 = f(t)$ and $y_2'(t) + p(t)y_2 = f(t)$ so $\frac{d}{dt}(y_1(t) - y_2(t)) + p(t)(y_1(t) - y_2(t)) = 0$.

(d) True, $f(y_0) = 0$.

Problem 2: (30 points) **Short Answer** for the following problems. No justification is needed.

(a) (10pts) Let $p(t)$ be a solution to the differential equation

$$p' = -r\left(1 - \frac{p}{L}\right)p; \qquad r, L > 0.$$

If $0 < p(0) < L$, what is $\lim_{t\to\infty} p(t)$?

(b) (10pts) Given the IVP

$$y' = -y^2; \qquad y(0) = 1$$

use Euler's method to create an approximate solution at $t = 1$ using a stepsize of $h = \frac{1}{2}$.

(c) (10pts) Consider the following differential equation

$$\frac{dy}{dt} = \frac{(1-y)^{(1/3)}}{(1-t)}.$$

What can you say about the existence and uniqueness of its solution, given an initial value of y(0) = 1?

Solution:

(a) $\boxed{\lim_{t\to\infty} p(t) = 0}$

(b) The solution is:

$$\begin{aligned} y_0 &= 1 \\ y_1 &= 1 + -(1)^2(1/2) = 1/2 \\ y_2 &= 1/2 - (1/2)^2(1/2) = \boxed{3/8} \end{aligned}$$

(c) Picard only indicates that $\boxed{\text{there exists a solution}}$ and does not provide any information about the uniqueness question.

Problem 3: (30 points)
Consider the differential equation

$$\frac{dy}{dt} = -\frac{e^t y^4}{3 + 3e^t} \,. \tag{1}$$

(a) (10 points) Find a solution to Eq. (1) by separation of variables.
(b) (5 points) Verify that your solution does indeed satisfy the differential equation.
(c) (7 points) Identify a solution that passes through the point $y(0) = 1$.
(d) (8 points) Find a solution to Eq. (1) that passes through the point $y(\sqrt{\pi}) = 0$.
Solution:

(a) Rewrite Eq. (1) as

$$3\frac{dy}{y^4} = -\frac{e^t dt}{1 + e^t} \,.$$

Integrating both sides gives

$$-\frac{1}{y^3} = -\ln|1 + e^t| + C \,.$$

Notice that $1 + e^t$ is always positive so we may drop the absolute value. Solving for $y(t)$ we find

$$y(t) = \frac{1}{(K + \ln(1 + e^t))^{1/3}} \,,$$

where $K = -C$.
(b) Notice that

$$y'(t) = -\frac{1}{3\,(K + \ln(1 + e^t))^{4/3}} \cdot \frac{e^t}{1 + e^t} = -\left(\frac{1}{(K + \ln(1 + e^t))^{1/3}}\right)^4 \frac{e^t}{3 + 3e^t} = -\frac{y^4 e^t}{3 + 3e^t} \,.$$

So this is indeed a solution.
(c) The solution that passes through $y(0) = 1$ corresponds to

$$1 = \frac{1}{(K + \ln(2))^{1/3}} \,.$$

Solving for K one obtains $K = 1 - \ln 2$. Hence the solution is given by

$$y(t) = \frac{1}{(1 + \ln(1 + e^t) - \ln 2)^{1/3}} = \frac{1}{\left(1 + \ln\left[\frac{1+e^t}{2}\right]\right)^{1/3}} \,.$$

(d) There is an equilibrium solution $y^* = 0$ to Eq. (1) that passes through the point $(\sqrt{\pi}, 0)$. One can see it by substituting $y = 0$ into the right hand side of Eq. (1).
Problem 4: (30 points)

(a) (10 points) Find all solutions of the homogeneous equation

$$y' + \frac{y}{t} = 0, \quad t > 0.$$

(b) (10 points) Find an integrating factor in order to obtain the particular solution of the equation

$$y' + \frac{y}{t} = f(t), \quad t > 0.$$

(c) (10 points) Using the integrating factor method, solve the initial value problem

$$\begin{cases} y' + \frac{y}{t} = \frac{\sin t}{t} \\ y\left(\frac{\pi}{2}\right) = 1. \end{cases}$$

Solution:

(a) We have

$$\frac{dy}{y} = -\frac{dt}{t}$$

so that

$$\log|y| = -\log|t| + \tilde{c},$$

where $\tilde{c}$ is a constant. We obtain

$$|y| = e^{\tilde{c}}\frac{1}{|t|}$$

and, since $t > 0$,

$$|y| = e^{\tilde{c}}\frac{1}{t}.$$

Replacing $e^{\tilde{c}}$ by an arbitrary real constant c, we arrive at the general solution of the homogeneous equation,

$$y(t) = \frac{c}{t}.$$

(b) Using equation for the integrating factor

$$\mu' = \frac{\mu}{t},$$

we have

$$\frac{d\mu}{\mu} = \frac{dt}{t}.$$

The same considerations as in the previous step yield

$$\mu(t) = \hat{c}t,$$

where $\hat{c}$ is a constant. Since for the integrating factor we need just one solution, we take $\hat{c} = 1$.

(c) Using the integrating factor t, we have

$$ty' + y = (ty)' = \sin t$$

and, thus,

$$ty = -\cos t + c.$$

Hence, the solution is

$$y(t) = -\frac{\cos t}{t} + \frac{c}{t}$$

and using the initial condition $y\left(\frac{\pi}{2}\right) = 1$, we obtain

$$y(t) = -\frac{\cos t}{t} + \frac{\pi}{2t}.$$

Problem 5: (30 points) 100 tons of the radioactive material Kryptonium are to be dumped in an underground deposit. The material undergoes radioactive decay and therefore the amount of Kryptonium at time t satisfies the equation $y' = ky$ if no additional Kryptonium is added or removed. The designers of the deposit want to know what specifications should the deposit have in the long-term. Your answers below may be left expressed in terms of elementary functions like $\ln(5)$, e^2, etc.

(a) (5 points) From laboratory experiments it is known that 9kg of Kryptonium decay to 3kg in 1 year. Using this information, find the value of the constant k.

(b) (10 points) Now assume that, after the initial deposit of 100 tons at time $t = 0$, Kryptonium is continuously dumped in the deposit at a rate of 10 tons per year. Write down a differential equation for the amount of Kryptonium $y(t)$ in the deposit.

(c) (10 points) Solve the differential equation you found in (b) and find $y(t)$.

(d) (5 points) Find the limiting amount of Kryptonium in the deposit as $t \to \infty$.

Solution:

(a) The amount of Kryptonium decays exponentially, so we know $y(t) = y(0)e^{kt}$. From the information given, $3 = 9e^k$. Solving, we find $k = \ln(\frac{1}{3}) = -\ln(3)$.

(b) In addition to decay, now we have the continuous addition of Kryptonium. The equation that models this is

$$y' = ky + 10 = -\ln(3)y + 10.$$

(c) The solution of the homogeneous equation is $y_h(t) = Ce^{-\ln(3)t}$. A particular solution is $y_p(t) = \frac{10}{\ln(3)}$. Therefore the general solution is $y(t) = y_p(t) + y_h(t) = \frac{10}{\ln(3)} + Ce^{-\ln(3)t}$. To find the constant, we note that $y(0) = 100$, so $100 = \frac{10}{\ln(3)} + C$, therefore $C = 100 - \frac{10}{\ln(3)}$ and

$$y(t) = \frac{10}{\ln(3)} + \left(100 - \frac{10}{\ln(3)}\right) e^{-\ln(3)t}.$$

(d) As $t \to \infty$, $y(t) \to \frac{10}{\ln(3)}$.