Problem 1: (30 points) **True/False** (answer True if it is always true otherwise answer False. No justification is needed.)

(a) (7pts) The equilibrium solution $y(t) \equiv 100$ of the logistic problem $y'(t) = 1/10(1 - y/100)y$ is unstable.

(b) (7pts) The differential equation $y''(t) + t^2y'(t) + e^ty(t) = t^3$ is nonlinear.

(c) (8pts) If $y_1(t) \neq y_2(t)$ are two solutions of $y'(t) + e^ty(t) = \sin^2 t$ then $y(t) = C(y_1(t) - y_2(t))$ with $C \in \mathbb{R}$ is the general solution of $y'(t) + e^ty(t) = 0$.

(d) (8pts) If $y_0(t) \equiv y_0$ is an equilibrium solution to $y'(t) = f(y)$, then $y_n = y_0$ when we use Euler’s method to solve the initial value problem $y'(t) = f(y)$, $y(0) = y_0$.

Solution:

(a) False, $f'(100) = -0.1 < 0$; or near $y = 100$, $f(y) < 0$ for $y > 100$ and $f(y) > 0$ for $y < 100$, it is stable.

(b) False.

(c) TRUE. $y_1(t) - y_2(t) \neq 0$ solves the homogenous equation since, $y_1'(t) + p(t)y_1 = f(t)$ and $y_2'(t) + p(t)y_2 = f(t)$ so $\frac{d}{dt} (y_1(t) - y_2(t)) + p(t) (y_1(t) - y_2(t)) = 0$.

(d) True, $f(y_0) = 0$.

Problem 2: (30 points) **Short Answer** for the following problems. No justification is needed.

(a) (10pts) Let $p(t)$ be a solution to the differential equation

$$p' = -r \left(1 - \frac{P}{L}\right) p; \quad r, L > 0.$$

If $0 < p(0) < L$, what is $\lim_{t \to \infty} p(t)$?

(b) (10pts) Given the IVP

$$y' = -y^2; \quad y(0) = 1$$

use Euler’s method to create an approximate solution at $t = 1$ using a stepsize of $h = \frac{1}{2}$.

(c) (10pts) Consider the following differential equation

$$\frac{dy}{dt} = \frac{(1 - y)^{1/3}}{(1-t)}.$$

What can you say about the existence and uniqueness of its solution, given an initial value of $y(0) = 1$?

Solution:
(a) \( \lim_{t \to \infty} p(t) = 0 \)

(b) The solution is:

\[
\begin{align*}
y_0 &= 1 \\
y_1 &= 1 + -\left(1/2\right)^2(1/2) = 1/2 \\
y_2 &= 1/2 - \left(1/2\right)^2(1/2) = 3/8
\end{align*}
\]

(c) Picard only indicates that there exists a solution and does not provide any information about the uniqueness question.

**Problem 3:** (30 points)
Consider the differential equation

\[
\frac{dy}{dt} = -\frac{e^t y^4}{3 + 3e^t}.
\]

(a) (10 points) Find a solution to Eq. (1) by separation of variables.

(b) (5 points) Verify that your solution does indeed satisfy the differential equation.

(c) (7 points) Identify a solution that passes through the point \( y(0) = 1 \).

(d) (8 points) Find a solution to Eq. (1) that passes through the point \( y(\sqrt{\pi}) = 0 \).

**Solution:**

(a) Rewrite Eq. (1) as

\[
3 \frac{dy}{y^4} = -\frac{e^t dt}{1 + e^t}.
\]

Integrating both sides gives

\[
-\frac{1}{y^3} = -\ln|1 + e^t| + C.
\]

Notice that \( 1 + e^t \) is always positive so we may drop the absolute value. Solving for \( y(t) \) we find

\[
y(t) = \frac{1}{(K + \ln(1 + e^t))^{1/3}},
\]

where \( K = -C \).

(b) Notice that

\[
y'(t) = -\frac{1}{3 (K + \ln(1 + e^t))^{4/3}} \cdot \frac{e^t}{1 + e^t} = -\left(\frac{1}{(K + \ln(1 + e^t))^{1/3}}\right)^4 \frac{e^t}{3 + 3e^t} = -\frac{y^4 e^t}{3 + 3e^t}.
\]

So this is indeed a solution.

(c) The solution that passes through \( y(0) = 1 \) corresponds to

\[
1 = \frac{1}{(K + \ln(2))^{1/3}}.
\]

Solving for \( K \) one obtains \( K = 1 - \ln 2 \). Hence the solution is given by

\[
y(t) = \frac{1}{(1 + \ln(1 + e^t) - \ln 2)^{1/3}} = \frac{1}{(1 + \ln \left[ \frac{1 + e^t}{2} \right])^{1/3}}.
\]

(d) There is an equilibrium solution \( y^* = 0 \) to Eq. (1) that passes through the point \( (\sqrt{\pi}, 0) \). One can see it by substituting \( y = 0 \) into the right hand side of Eq. (1).

**Problem 4:** (30 points)

(a) (10 points) Find all solutions of the homogeneous equation

\[
y' + \frac{y}{t} = 0, \quad t > 0.
\]
(b) (10 points) Find an integrating factor in order to obtain the particular solution of the equation
\[ y' + \frac{y}{t} = f(t), \quad t > 0. \]

(c) (10 points) Using the integrating factor method, solve the initial value problem
\[
\begin{cases}
  y' + \frac{y}{t} = \frac{\sin t}{t} \\
  y \left( \frac{\pi}{2} \right) = 1.
\end{cases}
\]

Solution:

(a) We have
\[
\frac{dy}{y} = -\frac{dt}{t}
\]
so that
\[
\log |y| = -\log |t| + \hat{c},
\]
where \( \hat{c} \) is a constant. We obtain
\[
|y| = e^{\hat{c}} \frac{1}{|t|}
\]
and, since \( t > 0 \),
\[
|y| = e^{\hat{c}} \frac{1}{t}.
\]
Replacing \( e^{\hat{c}} \) by an arbitrary real constant \( c \), we arrive at the general solution of the homogeneous equation,
\[
y(t) = \frac{c}{t}.
\]

(b) Using equation for the integrating factor
\[
\mu' = \frac{\mu}{t},
\]
we have
\[
\frac{d\mu}{\mu} = \frac{dt}{t},
\]
The same considerations as in the previous step yield
\[
\mu(t) = \hat{c}t,
\]
where \( \hat{c} \) is a constant. Since for the integrating factor we need just one solution, we take \( \hat{c} = 1 \).

(c) Using the integrating factor \( t \), we have
\[
(ty)' + y = (ty)' = \sin t
\]
and, thus,
\[
ty = -\cos t + c.
\]
Hence, the solution is
\[
y(t) = -\frac{\cos t}{t} + \frac{c}{t}
\]
and using the initial condition \( y \left( \frac{\pi}{2} \right) = 1 \), we obtain
\[
y(t) = -\frac{\cos t}{t} + \frac{\pi}{2t}.
\]
**Problem 5:** (30 points) 100 tons of the radioactive material Kryptonium are to be dumped in an underground deposit. The material undergoes radioactive decay and therefore the amount of Kryptonium at time \( t \) satisfies the equation \( y' = ky \) if no additional Kryptonium is added or removed. The designers of the deposit want to know what specifications should the deposit have in the long-term. Your answers below may be left expressed in terms of elementary functions like \( \ln(5) \), \( e^2 \), etc.

(a) (5 points) From laboratory experiments it is known that 9kg of Kryptonium decay to 3kg in 1 year. Using this information, find the value of the constant \( k \).

(b) (10 points) Now assume that, after the initial deposit of 100 tons at time \( t = 0 \), Kryptonium is continuously dumped in the deposit at a rate of 10 tons per year. Write down a differential equation for the amount of Kryptonium \( y(t) \) in the deposit.

(c) (10 points) Solve the differential equation you found in (b) and find \( y(t) \).

(d) (5 points) Find the limiting amount of Kryptonium in the deposit as \( t \to \infty \).

**Solution:**

(a) The amount of Kryptonium decays exponentially, so we know \( y(t) = y(0)e^{kt} \). From the information given, \( 3 = 9e^k \). Solving, we find \( k = \ln(\frac{1}{3}) = -\ln(3) \).

(b) In addition to decay, now we have the continuous addition of Kryptonium. The equation that models this is 
\[
y' = ky + 10 = -\ln(3)y + 10.
\]

(c) The solution of the homogeneous equation is \( y_h(t) = Ce^{-\ln(3)t} \). A particular solution is \( y_p(t) = \frac{10}{\ln(3)} \). Therefore the general solution is 
\[
y(t) = y_p(t) + y_h(t) = \frac{10}{\ln(3)} + Ce^{-\ln(3)t}.
\]
To find the constant, we note that \( y(0) = 100 \), so 
\[
100 = \frac{10}{\ln(3)} + C,
\]
and
\[
y(t) = \frac{10}{\ln(3)} + \left(100 - \frac{10}{\ln(3)}\right)e^{-\ln(3)t}.
\]

(d) As \( t \to \infty \), \( y(t) \to \frac{10}{\ln(3)} \).