Problem 1: (30 points) True/False (answer True if it is always true otherwise answer False. No justification is needed.)

(a) (7pts) The equilibrium solution $y(t) \equiv 100$ of the logistic problem $y'(t) = 1/10(1 - \frac{y}{100})y$ is unstable.

(b) (7pts) The differential equation $y''(t) + t^2 y'(t) + e^t y(t) = t^3$ is nonlinear.

(c) (8pts) If $y_1(t) \neq y_2(t)$ are two solutions of $y'(t) + e^t y(t) = \sin^2 t$ then $y(t) = C(y_1(t) - y_2(t))$ with $C \in \mathbb{R}$ is the general solution of $y'(t) + e^t y(t) = 0$.

(d) (8pts) If $y(t) \equiv y_0$ is an equilibrium solution to $y'(t) = f(y)$, then $y_n = y_0$ when we use Euler’s method to solve the initial value problem $y'(t) = f(y)$, $y(0) = y_0$.

Problem 2: (30 points) Short Answer for the following problems. No justification is needed.

(a) (10pts) Let $p(t)$ be a solution to the differential equation

$$p' = -r \left(1 - \frac{P}{L}\right) p; \quad r, L > 0.$$ 

If $0 < p(0) < L$, what is $\lim_{t \to \infty} p(t)$?

(b) (10pts) Given the IVP

$$y' = -y^2; \quad y(0) = 1$$

use Euler’s method to create an approximate solution at $t = 1$ using a stepsize of $h = \frac{1}{2}$.

(c) (10pts) Consider the following differential equation

$$\frac{dy}{dt} = \frac{(1 - y)^{(1/3)}}{(1 - t)}.$$ 

What can you say about the existence and uniqueness of its solution, given an initial value of $y(0) = 1$?

Problem 3: (30 points)

Consider the differential equation

$$\frac{dy}{dt} = -\frac{e^t y^4}{3 + 3e^t}.$$  

(a) (10 points) Find a solution to Eq. (1) by separation of variables.

(b) (5 points) Verify that your solution does indeed satisfy the differential equation.

(c) (7 points) Identify a solution that passes through the point $y(0) = 1$.

(d) (8 points) Find a solution to Eq. (1) that passes through the point $y(\sqrt{\pi}) = 0$.

See Problems on the Other Side!!!
Problem 4: (30 points)
(a) (10 points) Find all solutions of the homogeneous equation
\[ y' + \frac{y}{t} = 0, \quad t > 0. \]
(b) (10 points) Find an integrating factor in order to obtain the particular solution of the equation
\[ y' + \frac{y}{t} = f(t), \quad t > 0. \]
(c) (10 points) Using the integrating factor method, solve the initial value problem
\[
\begin{cases}
  y' + \frac{y}{t} = \frac{\sin t}{t} \\
  y\left(\frac{\pi}{2}\right) = 1.
\end{cases}
\]

Problem 5: (30 points) 100 tons of the radioactive material Kryptonium are to be dumped in an underground deposit. The material undergoes radioactive decay and therefore the amount of Kryptonium at time \( t \) satisfies the equation \( y' = ky \) if no additional Kryptonium is added or removed. The designers of the deposit want to know what specifications should the deposit have in the long-term. Your answers below may be left expressed in terms of elementary functions like \( \ln(5), e^2 \), etc.
(a) (5 points) From laboratory experiments it is known that 9kg of Kryptonium decay to 3kg in 1 year. Using this information, find the value of the constant \( k \).
(b) (10 points) Now assume that, after the initial deposit of 100 tons at time \( t = 0 \), Kryptonium is continuously dumped in the deposit at a rate of 10 tons per year. Write down a differential equation for the amount of Kryptonium \( y(t) \) in the deposit.
(c) (10 points) Solve the differential equation you found in (b) and find \( y(t) \).
(d) (5 points) Find the limiting amount of Kryptonium in the deposit as \( t \to \infty \).