1. (30 points) The following TRUE/FALSE questions are unrelated. If the statement is always true write TRUE, otherwise write FALSE. Supporting work or justification is not necessary and will not be considered or graded.

(a) An eigenvalue with algebraic multiplicity three has an eigenspace with dimension three.
(b) Consider the $n \times n$ matrix $A$. If $Ax = 0$ has only the solution $x = 0$, then $\lambda = 0$ is not an eigenvalue of $A$.
(c) The general solution of $y'' - 2y' + y = x^2$ is given by $y(x) = C_1 e^x + C_2 xe^x + x^2 + 4x + 1$.
(d) The functions $(1-x), (1+x), \text{ and } (1-3x)$ form a linearly independent set of solutions to $y'' = 0$ on $\mathbb{R}$.
(e) If a $2 \times 2$ matrix $A$ has an eigenvalue 0 with multiplicity two, and the associated eigenspace is one-dimensional, then the solutions to the linear system $x' = Ax$ are always bounded.
(f) If $x_1(t)$ and $x_2(t)$ are solutions of $x'(t) + p(t)x = f(t)$, then $x(t) = x_1(t) + C (x_1(t) - x_2(t))$ with $C \in \mathbb{R}$ is the general solution of $x'(t) + p(t)x = f(t)$.

2. (40 points) The following SHORT ANSWER questions are unrelated. For the questions in this problem, no motivation or supporting work is required. If you do submit supporting work, then [box] your answer, and know that your supporting work will not be graded.

(a) Find the general solution to $y + (1 + t)y = 0$.
(b) Determine a particular solution to the ODE $y'' + y = \sin(x)$ using the method of undetermined coefficients.
(c) What is the largest interval in $t$ on which the initial value problem

$$\sin(\pi t)y''(t) + \exp(t)y'(t) + \cos^{-1}(\pi t)y(t) = 0$$

with $y(0.8) = 0$, and $y'(0.8) = 1$ is guaranteed to have a unique solution?
(d) Suppose a $2 \times 2$ matrix $A$ has an eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with an associated eigenvalue $\lambda_1 = 3$, and an eigenvector $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with an associated eigenvalue $\lambda_2 = -2$. Compute $Av_3$ where $v_3 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.
(e) Determine and classify all equilibrium solutions for the differential equation $y' = -3(1-y)y$.
(f) For what initial conditions $y(t_0) = y_0$ is the ODE $y' = y\sqrt{T-y}$ guaranteed to have a unique solution?

3. (20 points) You have a large, usually happy, colony of European honey bees ($Apis mellifera$) in your backyard. Let $p(t)$ represent the colony population as a function of time, $t$, where time is measured in days. For each part below, write down a differential equation for $p(t)$ that models the dynamics of the population under the assumptions stated. Be sure to identify any restrictions on parameters that you introduce. The description in each part is independent of the other parts. Note: you are not being asked to solve the differential equations.

(a) Suppose that the increase in the bee population is directly proportional to the current number of bees.
(b) Suppose that the bee population always doubles every 10 days.
(c) Suppose that the number of bees always tends toward the carrying capacity $K > 0$ that the environment can sustain indefinitely.
(d) Suppose that the bee colony will become extinct if the population drops below the threshold value, $T$, where $T > 0$.

4. (20 points) Consider the ODE $y' = -6(y + 1)^{2/3}(3t^2 + 1)$.

(a) Find the general solution for $y > -1$.
(b) Now, solve the initial value problem if $y(-1) = 7$.
5. (30 points) Consider the linear system \( Ax = b \) where
\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & k & 3 \\
3 & 5 & 7
\end{pmatrix}
\quad \text{and} \quad
b = \begin{pmatrix}
0 \\
0 \\
p
\end{pmatrix}.
\]

(a) For what value(s) of \( k \) and \( p \) does this equation have

i. A unique solution?

ii. Infinite number of solutions?

iii. No solution?

(b) Compute the determinant of \( A \). For what value(s) of \( k \) is \( A \) non-invertible?

(c) For what value(s) of \( k \) and \( p \) is the vector \( b \) NOT in the span of the columns of \( A \)? For what value(s) of \( k \) are the columns of \( A \) linearly independent?

6. (30 points) A point mass \( m \) is suspended from massless string of length \( L > 0 \). Let \( \theta(t) \) be the angle of swing measured from vertical at time \( t \). Also, let \( g > 0 \) be the acceleration of gravity. When at rest, the mass hangs straight down in which case \( \theta = 0 \). If disturbed from its vertical resting position, the pendulum will oscillate. For small angles of swing (\( |\theta| \ll 1 \)) the motion of the pendulum can be approximated by the ODE
\[
L \ddot{\theta} + g\theta = 0.
\]

(a) Determine the general solution of the second order ODE.

(b) Write an equivalent coupled system of linear first–order ODEs equivalent to (1).

(c) Derive the general solution to the linear system in part (b).

(d) Identify and classify any equilibrium solution(s) of your linear system in part (b).

7. (30 points) Consider the coupled nonlinear system of ODEs for \( x(t) \) and \( y(t) \)
\[
\frac{dx}{dt} = x + x^2 - 3xy \\
\frac{dy}{dt} = 3y - y^2 - xy.
\]

(a) Determine the \((x(t), y(t))\) values of any equilibrium solution(s).

(b) Consider the equilibrium solution farthest from the origin in a phase portrait. Determine the linearization of the original nonlinear system at this location.

(c) Classify the equilibrium solution found in part (b).