ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section, (4) your instructor’s name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page, single-sided crib sheet is allowed.

Problem 1: (20 points) This problem will be graded solely on the answers provided. Motivations/derivations will not be considered.

(a) (5 points) For which of the parameters $b$ and $k$ does the differential equation
\[ y''(t) + by'(t) + ky(t) = 0 \]  
describe underdamped oscillations? Recall that an oscillator is underdamped if, as $t \to \infty$, $y(t)$ crosses 0 infinitely many times as it approaches $y = 0$. 
(i) $b = 4, k = 1$;
(ii) $b = 1, k = 1$;
(iii) $b = -1, k = 1$;
(iv) $b = 2, k = 2$.

(b) (5 points) Consider the family of matrices
\[ A = \begin{bmatrix} 1 & 1 \\ t & 2 \end{bmatrix}, \quad t \in \mathbb{R}. \]
For which values of $t$ does $A$
(i) have distinct, real eigenvalues?
(ii) have a double eigenvalue?
(iii) have complex eigenvalues with nonzero imaginary part?

(c) (4 points) Is the set of solutions to the differential equation
\[ y''(t) + t^2y(t) = t^3y'(t) + ty'''(t) \]
a vector space? If yes, what is its dimension?

(d) (6 points) What are the eigenvalues of the following matrices?
(i) \[ \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}; \]
(ii) \[ \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}; \]
(iii) \[ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \]

Problem 2: (20 points) Consider the differential equation
\[ y'' - \frac{2}{t^2} y = -3, \]
for $t \geq 1$ with homogeneous solutions $y_1(t) = t^2$, $y_2(t) = 1/t$. Use the method of variation of parameters to find the general solution to the nonhomogeneous problem.
Problem 3: (20 points) Consider the differential equation
\[ y'' + 4y' + 5y = e^{-2t}. \]
(a) (10 points) Find the general solution to the corresponding homogeneous equation.
(b) (10 points) Use the method of undetermined coefficients to find a particular solution to the nonhomogeneous problem.

Problem 4: (20 points)
(a) (16 points) Solve the initial value problem
\[
\begin{cases}
\ddot{y} + 9y = \sin(3t), \\
y(0) = 0, \\
\dot{y}(0) = 1.
\end{cases}
\]
(b) (4 points) Find one solution to the differential equation
\[ \ddot{y} + 9y = \sin(3t - 1). \]
*Hint: There is a relatively quick way to solve this Problem 4(b). Be mindful that this is worth 4 points and could be a lot of work.*

Problem 5: (20 points) Find all eigenvalues and eigenvectors to the following two matrices
\[
A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.
\]