ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section, (4) your instructor’s name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page, single-sided crib sheet is allowed.

Problem 1: (20 points) For the questions in this problem, no motivation is required. If you do submit work, then box your answer, and know that your work will not be graded.

(a) (5p) Let $A$ and $B$ be $3 \times 3$ matrices, and consider the following four matrices:

- $C = A^2 + 2AB + B^2$,
- $D = A(A + B) + B(B + A)$,
- $E = (A + B)(B + A)$,
- $F = A^2 + AB + BA + B^2$.

Which of these matrices must necessarily equal $(A + B)^2$?

(b) (5p) Let $k$ be a real number, and consider the matrix

$$A = \begin{pmatrix} k & 2 \\ 6 & k - 1 \end{pmatrix}.$$  

For which $k$ is the rank of $A$ less than 2?

(c) (5p) Let $v_1, v_2, \text{ and } v_3$ be three vectors in $\mathbb{R}^3$. You know that $V = \text{span}(v_1, v_2)$ is a plane through the origin (its dimension is two), and that $W = \text{span}(v_3)$ is a line through the origin. Then $\text{span}(v_1, v_2, v_3)$ is the smallest vector space containing both $V$ and $W$. What are the possible values of the dimension of $\text{span}(v_1, v_2, v_3)$?

(d) (5p) Let $V$ denote the vector space of continuous functions on an interval $[-1, 1]$. Which of the following subsets of $V$ are vector spaces?

(i) The set $W_1$ of all even functions in $V$ (in other words, if $f \in W_1$, then $f(x) = f(-x)$).
(ii) The set $W_2$ of all odd functions in $V$ (in other words, if $f \in W_2$, then $f(x) = -f(-x)$).
(iii) The set $W_3$ of all functions $f$ in $V$ such that $f(1) = 0$.
(iv) The set $W_4$ of all functions $f$ in $V$ such that $f(0) = 1$.

Problem 2: (20 points) Compute the determinant and the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$  

Problem 3: (20 points) Find all solutions to the linear system

- $x_1 + 2x_2 + x_3 + 3x_4 = 1$,
- $x_1 + 2x_2 + 2x_3 + 5x_4 = 0$,
- $-x_1 - 2x_2 - 3x_3 - 7x_4 = 1$. 

Problem 4: (20 points) Consider the matrices
\[
A = \begin{bmatrix}
0 & 2 & 1 & 2 \\
1 & -4 & -1 & -3 \\
-1 & 6 & 2 & 5
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
0 & 2 \\
1 & -4 \\
-1 & 6
\end{bmatrix}.
\]

*Hint:* The matrix \( A \) is row-equivalent to the matrix \( C = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1/2 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \).

(a) (8p) Which of the following four sets are linearly independent:

(i) The columns of \( A \).
(ii) The columns of \( B \).
(iii) The rows of \( A \).
(iv) The set \( \left\{ \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} \right\} \) consisting of the last three columns of \( A \).

(b) (8p) Recall that \( \text{col}(A) = \{ y \in \mathbb{R}^3 \text{ such that } y = Ax \text{ for some } x \in \mathbb{R}^4 \} \) is the span of the column vectors of \( A \), and that \( \text{null}(A) = \{ x \in \mathbb{R}^4 \text{ such that } Ax = 0 \} \) is the set of all solutions to \( Ax = 0 \). What are the dimensions of \( \text{col}(A) \) and \( \text{null}(A) \)?

(c) (4p) Specify a basis for \( \text{null}(A) \).

Problem 5: (20 points) For the system of differential equations
\[
\frac{dx}{dt} = x (y - 1) \\
\frac{dy}{dt} = y (y + x^2 - 2),
\]
provide the following qualitative analysis in the phase plane for \( x \) and \( y \) in the region \( 0 \leq x \leq 2 \) and \( 0 \leq y \leq 2 \):

(a) (4p) Determine and sketch the nullclines for this system.

(b) (4p) What are the equilibrium solutions?

(c) (4p) Show the direction of trajectories on the nullclines.

(d) (4p) Based on your phase plane analysis, classify the stability or instability of the equilibrium solutions.

(e) (4p) Specify the long term behavior of the solution starting at \( x_0 = 0.5 \) and \( y_0 = 0.5 \).