ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) recitation section (4) your instructor’s name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page one-sided crib sheet is allowed.

Problems 1d and 4 are given a small number of points in relation to the amount of work involved. You may want to save these for last.

Problem 1: (28p) These questions are worth 7 points each. No justification is necessary. If you do submit your work, then please box your final answer and know that the work will not be graded.

(a) Which of the following equations are linear differential equations in $y = y(t)$:

(i) $d^2y/dt^2 + 2 \sin(t) \frac{dy}{dt} + y = 0$.
(ii) $d^2y/dt^2 + 2 \frac{dy}{dt} + y - \sin(t) = 0$.
(iii) $d^2y/dt^2 + 2 \frac{dy}{dt} + \sin(y) = 0$.
(iv) $d^2y/dt^2 + 2 \sin(y) \frac{dy}{dt} + y = 0$.

(b) Consider the differential equation

$$\frac{dy}{dt} = \frac{1}{(t + y^2)},$$

$y(0) = 2$.

Specify the approximation to $y(1/2)$ that you obtain from a single step of the Euler method.

(c) Specify the general solution to

$$y''(t) + 4y(t) = 1.$$  

Hint: The general solution to the equation $z''(t) + 4z(t) = 0$ is $z(t) = A \cos(2t) + B \sin(2t)$, where $A$ and $B$ are arbitrary constants.

(d) Find one solution to

$$y'(t) \cos(t) - y(t) \sin(t) = \cos(2t).$$

Problem 2: (20p) Consider the differential equation

$$\frac{dy}{dx} = x y^2 e^{-x^2}. \quad (1)$$

(a) (5p) For what initial conditions $y(x_0) = y_0$ does a (possibly local) solution exist? Why?

(b) (5p) If the solution exists, is it unique? Why or why not?

(c) (10p) Solve the initial value problem for eq. (1) with $y(0) = 4$.

Problem 3: (20p) Consider a country whose population at time $t$ is $A(t)$ for $t \geq 0$. At time $t = 0$, you know that the population is precisely one million. Each year, the mortality rate is twenty out of every thousand, and the birth rate is ten for every thousand. The country also experiences a net migration of $s$ people every year ($s$ can be either positive or negative).

(a) (8p) Write down a differential equation that determines $A(t)$.

(b) (6p) Solve the differential equation.

(c) (6p) Specify the long term behavior of the population of the country for different values of $s$.  


Problem 4: (20p) Consider the direction field for a differential equation \( \frac{dy}{dt} = f(t, y) \). For all questions pertaining to this problem, utilize only the information pictured in this portion of the direction field.

(a) (3p) Based on the domain and range pictured, is the DE autonomous? separable? linear?

(b) (7p) Identify any equilibria of the DE and classify their stability.

(c) (7p) Given an initial condition \( y(0) = y_0 \in [-1.5, 1] \), determine the long term behavior of the solution by assuming that the direction field pictured extends unchanged for all \( t \). Your answer will depend on \( y_0 \).

(d) (3p) Write down a DE that exhibits the properties identified in (a)-(c) and could correspond to the direction field given. There are many possible choices.

Problem 5: (12p) Find the general solution to the equation

\[
y'(t) + \frac{y(t)}{t} = \frac{1}{\sqrt{t y(t)}}
\]

in the regime \( t > 0 \) and \( y > 0 \). Hint: Try to change the dependent variable to \( v(t) = (y(t))^{3/2} \).