1. (30 pts) Consider the function \( g(x, y, z) = \frac{4}{49} x^2 - \frac{1}{4} y^2 - z \). We want to determine the upward flux of the vector field \( F = -\frac{49}{8} i + 4j + k \) through that portion of the level surface \( g(x, y, z) = -1 \) lying inside \( \frac{4}{49} x^2 + \frac{1}{4} y^2 = 1 \).

(a) What type of quadric surface is \( g(x, y, z) = -1 \)?

**Solution:**

\[ g(x, y, z) = -1 = \frac{4}{49} x^2 - \frac{1}{4} y^2 - z \implies z = 1 + \frac{4}{49} x^2 - \frac{1}{4} y^2 \text{ which is a hyperbolic paraboloid.} \]

(It actually is a model of a Pringles® Chip).

(b) Using an inequality, describe the region \( R \) of the \( xy \)-plane over which you will need to integrate to calculate the requested flux.

**Solution:**

\[ \frac{4}{49} x^2 + \frac{1}{4} y^2 \leq 1 \]

(c) Find the integrand you will use to compute the requested flux.

**Solution:**

\[ g(x, y, z) = \frac{4}{49} x^2 - \frac{1}{4} y^2 - z \implies \nabla g = \left( \frac{8}{49} x, -\frac{1}{2} y, -1 \right) \text{ upward flux so use } -\nabla g \]

\[ p = k \implies |\nabla g \cdot p| = 1 \text{ and } F \cdot (-\nabla g) = \left\langle -\frac{49}{8}, 4, 1 \right\rangle \cdot \left\langle -\frac{8}{49} x, \frac{1}{2} y, 1 \right\rangle = x + 2y + 1 \]

\[ \iiint_S F \cdot dS = \iiint_R F \cdot |\nabla g \cdot p| \, dA = \iiint_R (x + 2y + 1) \, dA = \iiint_R (x + 2y + 1) \, dA \]

(d) The region \( R \) is kind of messy when it comes to actually computing the flux so let’s consider the change of variables \( u = \frac{2x}{7} \) and \( v = \frac{y}{2} \). Write an inequality describing the region in the \( uv \)-plane over which the integration will occur under this transformation.

**Solution:**

\[ \frac{4}{49} x^2 + \frac{1}{4} y^2 \leq 1 \text{ becomes } u^2 + v^2 \leq 1 \]

(e) Set up the integral using the order \( dv \, du \) to compute the flux. Don’t evaluate it...yet.

**Solution:**

\[ x = \frac{7}{2} u \text{ and } y = 2v \implies \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 7/2 & 0 \\ 0 & 2 \end{vmatrix} = 7 \text{ and } x + 2y + 1 = \frac{7}{2} u + 4v + 1 \]

\[ \iiint_R (x + 2y + 1) \, dA = \iiint_{u^2 + v^2 \leq 1} (x + 2y + 1) \, dA = \iiint_{u^2 + v^2 \leq 1} \left( \frac{7}{2} u + 4v + 1 \right) \, 7 \, dv \, du \]

\[ = \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \left( \frac{7}{2} u + 4v + 1 \right) \, 7 \, dv \, du \]
2. (25 pts) A friend of mine has been wandering aimlessly about \( \mathbb{R}^2 \) starting at the point \((1,0)\) and ending at the point \((0,1)\) in the presence of the force field

\[
F = \langle Axy - By^3, 4y + 3x^2 - 3xy^2 \rangle
\]

noting that no matter what path is taken between these two points the amount of work done is always the same. This nomadic life has been going on a very long time and my friend assures me that every path between the two points has been taken.

(a) What are \( A \) and \( B \)? Briefly explain.

**SOLUTION:**

Since we are told that the work done in the presence of \( F \) is independent of the path taken, \( F \) is conservative, implying that \( \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \). Now

\[
\frac{\partial P}{\partial y} = Ax - 3By^2 \quad \text{and} \quad \frac{\partial Q}{\partial x} = 6x - 3y^2 \implies A = 6, B = 1
\]

(b) Being in an adventurous yet mathematical mood, I want to know how much work I will do if I walk from \((-1,0)\) to \((3,-2)\) along the path \( y = \sqrt{x + 1}(x - 2)^{300} (x - 4)^{301} \). Can you please tell me?

**SOLUTION:**

Since \( F = \langle 6xy - y^3, 4y + 3x^2 - 3xy^2 \rangle \) is conservative, a potential function \( f \) exists such that \( F = \nabla f \). (Besides, you don’t want to parameterize the new path!)

\[
\frac{\partial f}{\partial x} = 6xy - y^3 \implies f(x, y) = \int (6xy - y^3) \, dx = 3x^2y - xy^3 + g(y)
\]

\[
\frac{\partial f}{\partial y} = 3x^2 - 3xy^2 + \frac{dg}{dy} = 4y + 3x^2 - 3xy^2 \implies \frac{dg}{dy} = 4y \implies g(y) = 2y^2 + c
\]

Thus \( f(x, y, z) = 3x^2y - xy^3 + 2y^2 + c \) and

\[
\text{Work} = \int_{(-1,0)}^{(3,-2)} F \cdot dr = f((3,-2)) - f((-1,0)) = 3(9)(-2) - 3(-8) + 2(4) + c - (0 + c) = -22
\]

3. (20 pts) Let \( F = \langle 3x + \cos y, 2y + \sin z, e^x + 5z \rangle \). Find the outward flux of \( F \) through the surface enclosing the region inside \( x^2 + z = 1 \), above the \( xy \)-plane and between \( y = 0 \) and \( y = 2 \).

**SOLUTION:**

The surface \( S \) and the region \( W \) it encloses satisfy the hypotheses of Gauss’ (Divergence) Theorem with

\[
\nabla \cdot F = \frac{\partial}{\partial x}(3x + \cos y) + \frac{\partial}{\partial y}(2y + \sin z) + \frac{\partial}{\partial z}(e^x + 5z) = 3 + 2 + 5 = 10
\]

\[
\iint_S F \cdot dS = \iiint_W \nabla \cdot F \, dV = \int_{-1}^{1} \int_{0}^{2} \int_{0}^{1-x^2} 10 \, dz \, dy \, dx = 10 \int_{-1}^{1} \int_{0}^{2} (1 - x^2) \, dy \, dx = 20 \int_{-1}^{1} (1 - x^2) \, dx = 40 \int_{0}^{1} (1 - x^2) \, dx = \frac{80}{3}
\]
4. (30 pts) Find the circulation of \( \mathbf{V}(x, y, z) = i + (x + yz)j + (xy - \cos^2 \sqrt{z}) k \) around the closed path composed of straight lines from \((1, 0, 0)\) to \((0, 0, 2)\) to \((0, 2, 0)\) to \((1, 0, 0)\) by evaluating an appropriate surface integral.

**SOLUTION:**

The closed path lies in the plane \( \frac{x}{1} + \frac{y}{2} + \frac{z}{2} = 1 \) or \( 2x + y + z = 2 \). This is the surface \( S \) of interest that we will project onto the \( xy \)-plane so that

\[
g(x, y, z) = 2x + y + z \quad \text{and} \quad \mathbf{p} = k \implies \nabla g = (2, 1, 1) \quad \text{and} \quad |\nabla g \cdot \mathbf{p}| = 1 \quad \text{(use } -\nabla g \text{ for proper orientation)}
\]

and \( \mathcal{R} = \{ (x, y) \in \mathbb{R}^2 | 0 \leq x \leq 1, 0 \leq y \leq -2x + 2 \} \).

\[
\nabla \times \mathbf{V} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x + yz & xy - \cos^2 \sqrt{z} \end{vmatrix} = (x - y)i - yj + k = (x - y, -y, 1)
\]

\[
\oint_C \mathbf{V} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{V} \cdot dS = \iint_{\mathcal{R}} (\nabla \times \mathbf{V}) \cdot \frac{-\nabla g}{|\nabla g \cdot \mathbf{p}|} \, dA = \iint_{\mathcal{R}} (x - y, -y, 1) \cdot (\frac{-2, -1, -1}{1}) \, dA = \int_0^1 \int_0^{1-x} (2x + 3y - 1) \, dy \, dx = \int_0^1 (10x^2 - 14x + 4) \, dx = \frac{1}{3}
\]

5. (30 pts) Evaluate \( \int_C y^2 \, dx + \left(x^3 + \sqrt{y^3 + 1}\right) \, dy \) where \( C \) is the counterclockwise path consisting of the bottom half of the unit circle with a rectangle of area 1 attached to the top.

**SOLUTION:**

Evaluating the line integral directly results in 4 integrals that are impossible/difficult to do, so we will use Green’s Theorem with \( \mathcal{R} \) being the region enclosed by \( C \). Application of the theorem can be simplified if we break \( \mathcal{R} \) into two pieces:

\[
\mathcal{R}_1 = \{ (x, y) \in \mathbb{R}^2 | -1 \leq x \leq 1, 0 \leq y \leq \frac{1}{2} \} \quad \text{(rectangle of area 1)}
\]

\[
\mathcal{R}_2 = \{ (x, y) \in \mathbb{R}^2 | -1 \leq x \leq 1, -\sqrt{1 - x^2} \leq y \leq 0 \} \quad \text{(bottom half of unit circle)}
\]

\[
P(x, y) = y^2 \implies \frac{\partial P}{\partial y} = 2y \quad \text{and} \quad Q(x, y) = x^3 + \sqrt{y^3 + 1} \implies \frac{\partial Q}{\partial x} = 3x^2
\]

Thus

\[
\oint_C y^2 \, dx + \left(x^3 + \sqrt{y^3 + 1}\right) \, dy = \iint_{\mathcal{R}_1} (3x^2 - 2y) \, dA + \iint_{\mathcal{R}_2} (3x^2 - 2y) \, dA
\]

\[
\iint_{\mathcal{R}_1} (3x^2 - 2y) \, dA = \int_{-1}^1 \int_0^{1/2} (3x^2 - 2y) \, dy \, dx = \int_{-1}^1 (3x^2y - y^3) \bigg|_0^{1/2} \, dx = \int_{-1}^1 \left( \frac{3}{2}x^2 - \frac{1}{4} \right) \, dx = 2 \int_0^1 \left( \frac{3}{2}x^2 - \frac{1}{4} \right) \, dx = 2 \left( \frac{1}{2}x^3 - \frac{1}{4}x \right) \bigg|_0^1 = \frac{1}{2}
\]

\[
\iint_{\mathcal{R}_2} (3x^2 - 2y) \, dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 (3x^2 - 2y) \, dy \, dx \quad \text{polar cords} \quad \int_{\pi}^{2\pi} \int_0^1 (3r^2 \cos^2 \theta - 2r \sin \theta) \, r \, dr \, d\theta
\]

\[
= \int_{\pi}^{2\pi} \int_0^1 (3r^3 \cos^2 \theta - 2r^2 \sin \theta) \, dr \, d\theta = \int_{\pi}^{2\pi} \left( \frac{3}{8} + \frac{3}{8} \cos 2\theta - \frac{2}{3} \sin \theta \right) \, d\theta
\]

\[
= \frac{3\pi}{8} + \frac{4}{3}
\]
Finally, then
\[ \int_C y^2 \, dx + (x^3 + \sqrt{y^3 + 1}) \, dy = \int_{R_1 \cup R_2} (3x^2 - 2y) \, dA = \frac{3\pi}{8} + \frac{4}{3} + \frac{1}{2} = \frac{3\pi}{8} + \frac{11}{6} \]

6. (15 pts) Daisy the Dachshund is a dog, so she loves to dig in the dirt. In a certain park, the satisfaction that Daisy gets from digging in the dirt at a point \((x, y)\) is given by the function

\[ S(x, y) = x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - 7y^2 \]

Assuming that the park has no boundary, can Daisy find a place in the park to maximize her dirt-digging satisfaction? If so, where is it and what is her maximum satisfaction? If she can’t find a place, explain why not.

**Solution:**

We’ll find the local extrema and classify them using the second derivative test. The first step is to find the critical points, that is, the points \((x, y)\) such that \(\nabla S(x, y) = 0\).

\[
\begin{align*}
S_x(x, y) &= 2x - x^2 - x^3 = -x(2 - x + x^2) = -x(x + 2)(x - 1) = 0, \\
S_y(x, y) &= -14y = 0.
\end{align*}
\]

This system gives us three critical points: \((-2, 0), (0, 0),\) and \((1, 0)\). Now we need to classify the critical points using second partial derivatives.

\[
\begin{align*}
S_{xx} &= 2 - 2x - 3x^2, \\
S_{yy} &= -14, \\
S_{xy} &= 0.
\end{align*}
\]

Also, 
\[ D(x, y) = S_{xx}S_{yy} - (S_{xy})^2 = -14S_{xx}. \]

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(D)</th>
<th>(S_{xx})</th>
<th>Type of Extrema or Saddle?</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-2, 0))</td>
<td>84</td>
<td>-6</td>
<td>Local Maximum</td>
</tr>
<tr>
<td>((0, 0))</td>
<td>-28</td>
<td>2</td>
<td>Saddle</td>
</tr>
<tr>
<td>((1, 0))</td>
<td>42</td>
<td>-3</td>
<td>Local Maximum</td>
</tr>
</tbody>
</table>

Which of the local maximum has a larger value? \(S(-2, 0) = \frac{8}{3}\) and \(S(1, 0) = \frac{5}{12}\). Daisy achieves maximum satisfaction when she digs in the dirt at the point \((-2, 0)\). Her maximum dirt-digging satisfaction is 
\[ S(-2, 0) = \frac{8}{3}. \]

Why is this a global maximum? Notice that the trace in the \(xz\)-plane is a downward facing 4th degree polynomial. Also, the trace in the \(yz\)-plane is the parabola \(z = -7y^2\). The surface is a decreasing function of \(x\) and \(y\) outside of the circle of radius 2 centered at the origin.