1. (15 pts) Indicate in your blue book which of the following statements are True and which are False. No explanation required.

(a) \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w} \)
(b) The cross product of two nonzero vectors that are scalar multiples of each other has magnitude 0.
(c) \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \times \mathbf{b}) \times \mathbf{a} \)
(d) The intersection of the plane \( y = 0 \) and the surface \( 4x^2 + y^2 + 4z^2 - 4y - 24z + 36 = 0 \) is an ellipse.
(e) If \( \mathbf{u}(t) \) and \( \mathbf{v}(t) \) are differentiable vector functions, then \( \frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}'(t) \)

**Solution:**

(a) (3 pts) **False:** The left side is defined, but the cross product of a scalar and a vector is not defined.

(b) (3 pts) **True:** \( ||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}|| ||\mathbf{b}|| \sin(0) = 0. \)

(c) (3 pts) **True:** \( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \).

(d) (3 pts) **False:** The intersection satisfies \( 4x^2 + 4(z - 3)^2 = 0 \) or \( (x, y, z) = (0, 0, 3) \) which is a point.

(e) (3 pts) **False:** Instead, \( \frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t) \).

2. (10 pts) An important parameter in aircraft takeoff and landing operations is the crosswind, the component of the wind perpendicular to a runway. Suppose takeoffs and landings on a particular runway must be suspended when the crosswind exceeds 15 mph. If the wind is blowing from the west at 36 mph, can aircraft use a runway directed N60°W? Justify your answer.

**Solution:**

**Method 1** The crosswind, \( \mathbf{c} \), is the orthogonal projection of the wind vector, \( \mathbf{w} \), onto the runway, \( \mathbf{r} \).

The unit vector along the runway is \( \mathbf{r} = \langle \cos(150°), \sin(150°) \rangle = \langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle \). The wind vector is \( 36\mathbf{i} = \langle 36, 0 \rangle \).

\[
\text{proj}_w \mathbf{w} = \frac{\mathbf{w} \cdot \mathbf{r}}{\mathbf{r} \cdot \mathbf{r}} \mathbf{r} = \left( \frac{36, 0}{\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle \cdot \langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle} \right) \left( \frac{-\sqrt{3}}{2}, \frac{1}{2} \right) = -18\sqrt{3} \left( \frac{-\sqrt{3}}{2}, \frac{1}{2} \right) = \langle 27, -9\sqrt{3} \rangle.
\]

\[
\text{orth}_w \mathbf{w} = \mathbf{w} - \text{proj}_w \mathbf{w} = \langle 36, 0 \rangle - \langle 27, -9\sqrt{3} \rangle = \langle 9, 9\sqrt{3} \rangle.
\]

\[
||\text{orth}_w \mathbf{w}|| = ||9 \langle 1, \sqrt{3} \rangle|| = 9 \cdot 2 = 18 \text{ mph}.
\]

The wind exceeds 15 mph, so the runway is unusable.

**Method 2** \( ||\mathbf{c}|| = ||\mathbf{w}|| \sin(30°) = 36(1/2) = 18 \) mph. The runway is unusable.

3. (15 pts) A particle travels along the helix given by \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k} \). At time \( t = \pi \) the particle leaves the path and flies off on a tangent. Find the location of the particle at \( t = 2\pi \) assuming no forces act on it after it leaves the helix.

**Solution:**

The particle will fly along a line in the direction of the tangent vector to the helix at \( t = \pi \), that is, \( \mathbf{r}'(\pi) \), that contains the point \( \mathbf{r}(\pi) \). The tangent vector is \( \mathbf{r}'(t) = (-\sin t, \cos t, 1) \). We can parametrize the line as

\[
\mathbf{r}_{\text{line}}(s) = \mathbf{r}(\pi) + s\mathbf{r}'(\pi) = (-1, 0, \pi) + s(0, -1, 1) = (-1, -s, \pi + s) \quad s \geq \pi
\]

When \( t = 2\pi, s = \pi \) and the particle has position vector \( \mathbf{r}_{\text{line}}(\pi) = (-1, -\pi, 2\pi) \).
4. (14 pts) Consider the vector function \( \mathbf{r}(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle \) for \( 0 \leq t \leq c \). Find the value of \( c \) for which the arc length is \( 2\sqrt{5} \).

**SOLUTION:**

The arc length function is

\[
 s(t) = \int_0^t \| \mathbf{r}'(u) \| \, du.
\]

We can calculate

\[
 \mathbf{r}'(t) = \langle 2t, \cos t - \cos t + \sin t + t \cos t, \cos t + \sin t - \cos t + t \sin t \rangle = \langle 2t, t \sin t, t \cos t \rangle.
\]

\[
 \| \mathbf{r}'(t) \| = \sqrt{4t^2 + t^2(\sin t)^2 + t^2(\cos t)^2} = \sqrt{5t^2} = \sqrt{5}t, \quad \text{for } t \geq 0.
\]

We want to solve

\[
 2\sqrt{5} = s(c) = \int_0^c \sqrt{5}t \, dt = \frac{\sqrt{5}}{2} t^2 \bigg|_0^c = \frac{\sqrt{5}}{2} c^2.
\]

We must have \( 4 = c^2 \) or \( c = \pm 2 \). Since \( 0 \leq t \leq c, \ c = 2 \).

5. (16 pts) Consider two glass plates (planes). The first plane, \( P_1 \), intersects the \( x \)-, \( y \)-, and \( z \)-axes at the locations \( (2,0,0) \), \( (0,2,0) \), and \( (0,0,3) \). The second plane, \( P_2 \), is parallel to the \( x \)-axis and intersects the remaining two axes at the same points as plane \( P_1 \).

(a) Determine the equation of plane \( P_1 \) and its unit normal vector \( \mathbf{n}_1 \).

(b) Determine the equation of plane \( P_2 \) and its unit normal vector \( \mathbf{n}_2 \).

(c) Determine the cosine of the angle between the glass plates.

(d) A laser beam at the origin is aimed perpendicular to plane \( P_1 \) and pierces the plane \( P_2 \) at point \( Q \). What are the coordinates of point \( Q \)?

**SOLUTION:**

(a) Since we know the points where the plane intersects the coordinate axes, the equation for \( P_1 \) is

\[
 \frac{x}{2} + \frac{y}{2} + \frac{z}{3} = 1 \implies 3x + 3y + 2z = 6
\]

Its unit normal is

\[
 \mathbf{n}_1 = \frac{\langle 3, 3, 2 \rangle}{\|\langle 3, 3, 2 \rangle\|} = \frac{\langle 3, 3, 2 \rangle}{\sqrt{9 + 9 + 4}} = \frac{1}{\sqrt{22}} \langle 3, 3, 2 \rangle
\]

(b) Two points in \( P_2 \) are \( A(0,2,0) \) and \( B(0,0,3) \). Since \( P_2 \) is parallel to the \( x \)-axis, another point in \( P_2 \) is \( C(2,0,3) \). The vector \( \mathbf{AB} \times \mathbf{AC} \) is normal to \( P_2 \).

\[
 \mathbf{AB} \times \mathbf{AC} = \begin{vmatrix}
 1 & j & k \\
 0 & -2 & 3 \\
 2 & -2 & 3
\end{vmatrix} = 6j + 4k
\]

The unit normal to \( P_2 \) is

\[
 \mathbf{n}_2 = \frac{\langle 0, 6, 4 \rangle}{\|\langle 0, 6, 4 \rangle\|} = \frac{\langle 0, 6, 4 \rangle}{\sqrt{0 + 36 + 16}} = \frac{\langle 0, 6, 4 \rangle}{\sqrt{52}} = \frac{1}{\sqrt{13}} \langle 0, 3, 2 \rangle
\]

The equation of \( P_2 \) is \( \mathbf{n}_2 \cdot \langle x - 0, y - 0, z - 3 \rangle = 3y + 2(z - 3) = 0 \) or

\[
 3y + 2z = 6
\]

(c)

\[
 \cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{9 + 4}{\sqrt{13}\sqrt{22}} = \frac{\sqrt{13}}{22}
\]
6. (15 pts) Two bugs fly along the space curves given by

\[ \mathbf{r}_1(t) = (t, t^2, t^3) \quad \text{and} \quad \mathbf{r}_2(t) = (1 + 2t, 1 + 6t, 1 + 14t) \quad -\infty < t < \infty \]

Determine the time(s) and position(s), if any, where the bugs collide and the time(s) and position(s), if any, where the bugs’ paths intersect.

**Solution:**
To collide, the bugs must occupy the same point at the same time. That is, \( \mathbf{r}_1(t) = \mathbf{r}_2(t) \) or \( (t, t^2, t^3) = (1 + 2t, 1 + 6t, 1 + 14t) \).

\[ t = 1 + 2t \Rightarrow t = -1. \]

However, this does not satisfy the other equations. The bugs do not collide.

The paths intersect if there exist \( t \) and \( s \) such that \( \mathbf{r}_1(t) = \mathbf{r}_2(s) \) or

\[ t = 1 + 2s, \]
\[ t^2 = 1 + 6s, \]
\[ t^3 = 1 + 14s. \]

Substitute the first equation into the second to yield

\[ (1 + 2s)^2 = 1 + 6s \Rightarrow 1 + 4s + 4s^2 = 1 + 6s \Rightarrow 4s^2 - 2s = 0 \Rightarrow 2s(2s - 1) = 0, \]

so \( s = 0 \) or \( s = \frac{1}{2} \).

If \( s = 0 \), \( t = 1 + 2(0) = 1 \) and \( \mathbf{r}_1(1) = \mathbf{r}_2(0) = (1, 1, 1) \), so one point of intersection is \( (1, 1, 1) \).

If \( s = \frac{1}{2}, t = 1 + 2 \left( \frac{1}{2} \right) = 2 \) and \( \mathbf{r}_1(2) = \mathbf{r}_2 \left( \frac{1}{2} \right) = (2, 4, 8) \), so one point of intersection is \( (2, 4, 8) \).

Note that these values of \( s \) and \( t \) also satisfy the third equation above.

7. (15 pts) Find the position vector \( \mathbf{r}(t) \) of an object subject to the following conditions: it undergoes an acceleration of \( e^t \mathbf{i} + 2t \mathbf{j} + (t + 1) \mathbf{k} \) for \( t \geq 0 \) and it begins its motion at \( 2 \mathbf{i} + \mathbf{j} + \mathbf{k} \) with a velocity of \( \mathbf{i} + \mathbf{k} \).

**Solution:**
Integrate the acceleration to find that

\[ \mathbf{v}(t) = \int \mathbf{a}(t) \, dt = \int \mathbf{r}''(t) \, dt = \mathbf{r}'(t) = \left\langle e^t, 2t, t + 1 \right\rangle \quad \Rightarrow \quad \mathbf{C} = \left\langle 0, 0, 1 \right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle e^t, t^2, \frac{1}{2} t^2 + t + 1 \right\rangle. \]

Integrate the velocity to find that

\[ \mathbf{r}(t) = \int \mathbf{v}(t) \, dt = \int \mathbf{r}'(t) \, dt = \left\langle e^t, t^2, \frac{1}{2} t^2 + t + 1 \right\rangle \quad \Rightarrow \quad \mathbf{C} = \left\langle 1, 1, 1 \right\rangle \quad \Rightarrow \quad \mathbf{r}(t) = \left\langle e^t + 1, \frac{1}{3} t^3 + 1, \frac{1}{6} t^3 + \frac{1}{2} t^2 + t + 1 \right\rangle. \]