INSTRUCTIONS: Electronic devices, books, and crib sheets are not permitted. Write your name and your instructor's name on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

Note: this is a Calculus III final exam. While some problems may be solved using techniques from Calculus I or II, you may not receive full credit if you do so, even if the final result is correct.

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1. (40 points) The dome of a planetarium is described by the function $z = \sqrt{900 - x^2 - y^2}$. To service the dome, a very small flat hatch measuring approximately 1 unit by 1 unit is installed tangent to the dome surface. The coordinates of the center of the hatch are $(10, 20, 20)$.

   (a) Determine the unit normal vector to the hatch.

   (b) Determine the standard equation of the plane defined by the hatch.

   (c) If the sun is directly overhead and the hatch is open, approximate the surface area of the patch of sunlight on the floor.

   (d) The air pressure inside the dome is greater than the surrounding atmospheric pressure. As a result, there exists an outward flux of air through the hatch. If the velocity of the air across the hatch is $\mathbf{v} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, estimate the flux of air across the open hatch.

   SOLUTION:

   (a) The planetarium dome is the level surface $g(x, y, z) = 900$ where $g(x, y, z) = x^2 + y^2 + z^2$. The normal to this surface is $\nabla g(x, y, z) = \langle 2x, 2y, 2z\rangle$ giving $\nabla g(10, 20, 20) = (20, 40, 40)$. Thus the unit normal to the hatch is

   $$\mathbf{n} = \pm\frac{\nabla g}{\|\nabla g\|} = \pm\frac{\langle 20, 40, 40\rangle}{\sqrt{20^2 + 40^2 + 40^2}} = \pm\frac{1}{3}\langle 1, 2, 2\rangle$$

   (b)

   $$\begin{aligned}\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0 &\implies \frac{1}{3}\langle 1, 2, 2\rangle \cdot (\langle x, y, z\rangle - \langle 10, 20, 20\rangle) = 0\\ &\implies (x - 10) + 2(y - 20) + 2(z - 20) = 0\\ &\implies x + 2y + 2z = 90\end{aligned}$$

   (c) The patch of sunlight on the floor is essentially the integration region, $\mathcal{R}$, that would be used to find the surface area of the hatch if it is projected onto the xy–plane. The equation of the hatch, the surface, is $g(x, y, z) = x + 2y + 2z$ with $\mathbf{p} = \mathbf{k}$. This gives

   $$\nabla g = \langle 1, 2, 2\rangle \implies \|\nabla g\| = 3 \quad\text{and}\quad |\nabla g\cdot\mathbf{p}| = 2$$

   Thus

   $$\text{Area of hatch} = \iint_{\text{hatch}} \mathrm{d}S = \iint_{\text{sun patch}} \frac{\|\nabla g\|}{|\nabla g\cdot\mathbf{p}|}\,\mathrm{d}A = \iint_{\text{sun patch}} \frac{3}{2}\,\mathrm{d}A$$

   $$\implies \iint_{\text{sun patch}} \mathrm{d}A = \frac{2}{3}\iint_{\text{hatch}} \mathrm{d}S = \frac{2}{3}\times\text{area(hatch)} = \frac{2}{3}(1\times 1) = \frac{2}{3}$$

   (d) We use the upward pointing normal to the hatch to obtain the outward flux of air through the hatch. Using the information from part (c), we have

   $$\begin{aligned}\text{Flux} = \iint_{\text{hatch}} \mathbf{v}\cdot\mathrm{d}\mathbf{S} &= \iint_{\text{sun patch}} \mathbf{v}\cdot\frac{\pm\nabla g}{|\nabla g\cdot\mathbf{p}|}\,\mathrm{d}A = \iint_{\text{sun patch}} \langle x, y, z\rangle\cdot\frac{\langle 1, 2, 2\rangle}{2}\,\mathrm{d}A\\ &= \iint_{\text{sun patch}} \frac{x + 2y + 2z}{2}\,\mathrm{d}A = \iint_{\text{sun patch}} \frac{90}{2}\,\mathrm{d}A \quad\text{(use surface to eliminate } z\text{)}\\ &= 45\iint_{\text{sun patch}} \mathrm{d}A = 45\times\text{area(sun patch)} = 45\left(\frac{2}{3}\right) = 30\end{aligned}$$

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2. (40 Points) On a given day, the temperature distribution (in degrees Fahrenheit) is given by $T(x, y) = 50 + xy + x^2$. Two turtles, Burt and Beatrice, leave their food bowl located at $(0, 0)$ and move off into the first quadrant. Then have had a little argument, so Burt travels along the path defined by $y = x^2$ and Beatrice travels along the path $y = \sqrt{x}$. Each one thinks they travel faster than the other, but they both travel at 2 meters per hour.

(a) Bert and Beatrice meet again along their paths. Where does this occur?

(b) Set up, but do not evaluate, the calculations necessary to determine when they meet.

(c) When they meet, at what rate is $T(x, y)$ changing with respect to *distance* for each turtle? Clearly identify your results.

(d) If the turtle experiencing the largest rate of increase in temperature with respect to *distance* travels for a short time $\Delta t = 0.1$, by approximately how much will the temperature change for that turtle?

(e) If the other turtle travels for a short distace $\Delta s = 0.1$, by approximtely how much will the temperature change for that turtle?

SOLUTION:

(a) Setting the paths equal to one another yields $x^2 = \sqrt{x} \implies x^4 - x = x(x^3 - 1) = 0 \implies x = 0, 1$ so the turtles meet again at $(x, y) = (1, 1)$.

(b) We know the turtles' speed to be 2 meters per hour. We can use the fact that time equals distance divided by speed to find the time each turtle reaches the meeting point, where the distance is the arc length of each curve for $0 \le x \le 1$ and the arc length is $s = \int_0^1 \sqrt{1 + [y'(x)]^2}\, \mathrm{d}x$.

$$t_{\text{Bert}} = \frac{1}{2}\int_0^1 \sqrt{1 + 4x^2}\, \mathrm{d}x \qquad t_{\text{Beatrice}} = \frac{1}{2}\int_0^1 \sqrt{1 + (1/4x)}\, \mathrm{d}x$$

The latter integral is improper but not really necessary as the arc lengths are both the same since the paths are simply inverses of one another. These formulas give the time in hours when the turtles meet again.

(c) We need to compute directional derivatives at each point, requiring the direction (provided by a unit vector) each turtle is moving at the meeting point as well as the gradient of $T(x, y)$ at $(x, y) = (1, 1)$ which is $\nabla T(1, 1) = \langle 2x + y, x\rangle\big|_{(1,1)} = \langle 3, 1\rangle$.

Burt:

Direction vector is found using the slope of the tangent line to his curve, $y' = 2x$, at $(1, 1)$ which is $\langle 1, 2\rangle \implies \|\langle 1, 2\rangle\| = \sqrt{5}$.

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \langle 3, 1\rangle \cdot \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \sqrt{5}$$

Beatrice:

Direction vector is found using the slope of the tangent line to her curve, $y' = 1/2\sqrt{x}$, at $(1, 1)$ which is $\langle 2, 1\rangle \Rightarrow \|\langle 2, 1\rangle\| = \sqrt{5}$.

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \langle 3, 1\rangle \cdot \frac{1}{\sqrt{5}}\langle 2, 1\rangle = \frac{7}{5}\sqrt{5}$$

(d)

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \implies \Delta T \approx \frac{\mathrm{d}T}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\Delta t = \frac{7\sqrt{5}}{5}(2)(0.1) = \frac{7}{25}\sqrt{5}$$

(e)

$$\Delta T \approx \frac{\mathrm{d}T}{\mathrm{d}s}\Delta s = \sqrt{5}(0.1) = \frac{\sqrt{5}}{10}$$

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3. (40 Points) The strength of a cell phone signal in a certain section of a city can be described by the function $f(x, y) = 9 - \frac{x^3}{3} - \frac{y^3}{3}$ where $x \ge 0$ and $y \ge 0$. A highway passes through the city along the path described by $xy = 16$. As you drive along the highway, at what location do you get the strongest cell phone signal? The more Calculus 3 concepts you use, the higher your grade!

SOLUTION:

We solve this problem using the method of Lagrange Multipliers. The function we wish to maximize (the objective function) is $f(x, y) = 9 - x^3/3 - y^3/3$ subject to the constraint $g(x, y) = xy = 16$.

$$\begin{aligned} f_x &= -x^2 \quad g_x = y \quad \implies -x^2 = \lambda y \\ f_y &= -y^2 \quad g_y = x \quad \implies -y^2 = \lambda x \end{aligned}$$

Since neither x nor y can be zero (otherwise the constraint won't be satisfied), we can solve each of the last equations for λ and equate the results to get

$$-\frac{x^2}{y} = -\frac{y^2}{x} \implies x^3 = y^3 \implies y = x$$

The constraint then becomes $x^2 = 16 \implies x = 4\ (x \ge 0) \implies y = 4$. Thus the critical point is $(x, y) = (4, 4)$. This critical point yields a maximum of the objective function. This follows from the fact that if x moves away from 4 in either direction along the constraint curve, the value of f will decrease. The strongest cell phone signal occurs at $(4, 4)$. ■

4. (40 Points) Consider the vector field $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + xyz\,\mathbf{k}$ and the finite object bounded on the bottom by the surface $z = x^2 + y^2$, and on the top by $z = 1$.

   (a) Calculate the outward flux of $\mathbf{F}$ across the **top** surface of the object.

   (b) Calculate the outward flux of $\mathbf{F}$ across the **bottom** surface of the object.

   (c) Determine the net outward flux over the surface of the object.

   (d) If possible, verify your "net outward flux" calculation using any theorem(s) from Calculus III, and clearly state your reasoning! Otherwise, clearly write "Cannot be verified".

SOLUTION:

(a) The top surface of the object, $\mathcal{S}_t$, is the plane $z = 1$ and we'll project this surface onto the xy–plane, yielding the unit disk as the region, $\mathcal{R}$, of integration. We have $\mathbf{p} = \mathbf{k}$ and

$$g(x,y,z) = z \implies \nabla g = \langle 0,0,1\rangle \text{ and } |\nabla g\cdot\mathbf{p}| = 1 \text{ and we choose } +\nabla g \text{ for the outward normal}$$

$$\mathbf{F}\cdot\frac{\nabla g}{|\nabla g\cdot\mathbf{p}|} = \langle -y, x, xyz\rangle\cdot\frac{\langle 0,0,1\rangle}{1} = xyz = xy(1) \text{ where we used the surface to eliminate } z$$

$$\iint_{\mathcal{S}_t}\mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint\limits_{x^2+y^2\le 1} xy\,\mathrm{d}A = \int_0^{2\pi}\int_0^1 (r\cos\theta)\,(r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{2}\left(\int_0^{2\pi}\sin 2\theta\,\mathrm{d}\theta\right)\left(\int_0^1 r^3\,\mathrm{d}r\right) = 0$$

(b) The bottom surface of the object, $\mathcal{S}_b$, is the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ and we'll project this surface onto the xy–plane, yielding the unit disk as the region, $\mathcal{R}$, of integration. We have $\mathbf{p} = \mathbf{k}$ and

$$g(x,y,z) = x^2 + y^2 - z \implies \nabla g = \langle 2x, 2y, -1\rangle \text{ and } |\nabla g\cdot\mathbf{p}| = 1 \text{ and we choose } +\nabla g \text{ for the outward normal}$$

$$\mathbf{F}\cdot\frac{\nabla g}{|\nabla g\cdot\mathbf{p}|} = \langle -y, x, xyz\rangle\cdot\frac{\langle 2x, 2y, -1\rangle}{1} = -xyz = -xy(x^2+y^2) \text{ where we used the surface to eliminate } z$$

$$\iint_{\mathcal{S}_b}\mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint\limits_{x^2+y^2\le 1} -xy(x^2+y^2)\,\mathrm{d}A = \int_0^{2\pi}\int_0^1 -(r\cos\theta)\,(r\sin\theta)\,r^3\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\left(\int_0^{2\pi}\sin 2\theta\,\mathrm{d}\theta\right)\left(\int_0^1 r^5\,\mathrm{d}r\right) = 0$$

(c) The entire surface of the object, $\mathcal{S}$ is given by $\mathcal{S} = \mathcal{S}_b \cup \mathcal{S}_t$. Thus the net outward flux over the entire surface of the object is

$$\iint_{\mathcal{S}}\mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint_{\mathcal{S}_b}\mathbf{F}\cdot\mathrm{d}\mathbf{S} + \iint_{\mathcal{S}_t}\mathbf{F}\cdot\mathrm{d}\mathbf{S} = 0 + 0 = 0$$

(d) The object, $\mathcal{E}$, and its boundary, $\mathcal{S}$, are candidates for Gauss's Divergence Theorem. We have $\nabla\cdot\mathbf{F} = xy$ where the object is described by the inequalities $x^2+y^2 \le z \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, and $-1 \le x \le 1$. Using cylindrical coordinates gives

$$\text{Flux} = \iint_{\mathcal{S}}\mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iiint_{\mathcal{E}}\boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_{r^2}^1 (r\cos\theta)(r\sin\theta)\,r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{2}\left(\int_0^{2\pi}\sin 2\theta\,\mathrm{d}\theta\right)\int_0^1\int_{r^2}^1 r^3\,\mathrm{d}z\,\mathrm{d}r = 0$$

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5. (40 Points) Consider the same vector field and object from the previous problem, and the clockwise path around the "top edge" of the object.

   (a) Calculate the circulation around this path.

   (b) If possible, verify your calculation in part (a) using any theorem(s) from Calculus III, and clearly state your reasoning! Otherwise, clearly write "Cannot be verified."

SOLUTION:

(a) Parameterize the path as $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k},\;\; 0 \le t \le 2\pi$, giving $\mathbf{r}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + 0\,\mathbf{k}$ and

$$\mathbf{F}(\mathbf{r}(t)) = -\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sin t\cos t\,\mathbf{k} \implies \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -1$$

so that the circulation is given as

$$\text{Circulation} = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} -1\,\mathrm{d}t = -2\pi$$

(b) We can verify the calculation using Stokes' Theorem with the surface $\mathcal{S}$ being the paraboloid $z = x^2 + y^2$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & xyz \end{vmatrix} = xz\,\mathbf{i} - yz\,\mathbf{j} + 2\,\mathbf{k}$$

We project the surface onto the xy–plane, yielding $\mathbf{p} = \mathbf{k}$, the integration region, $\mathcal{R}$, the unit disk, and

$$g(x, y, z) = x^2 + y^2 - z \implies \nabla g = \langle 2x, 2y, -1 \rangle \implies |\nabla g \cdot \mathbf{p}| = |-1| = 1$$

Based on the orientation of the path, we use $+\nabla g$ for the normal to the surface giving

$$(\nabla \times \mathbf{F}) \cdot \frac{+\nabla g}{|\nabla g \cdot \mathbf{p}|} = \langle xz, -yz, 2 \rangle \cdot \langle 2x, 2y, -1 \rangle = 2x^2 z - 2y^2 z - 2 = 2z(x^2 - y^2) - 2$$

and, using the surface to eliminate z, we obtain

$$\text{Circulation} = \iint\limits_{x^2+y^2 \le 1} \left[2(x^2 + y^2)(x^2 - y^2) - 2\right] \mathrm{d}A = 2\int_0^{2\pi}\int_0^1 \left[r^2\left(r^2\cos^2\theta - r^2\sin^2\theta\right) - 1\right] r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 2\int_0^{2\pi}\int_0^1 \left(r^5\cos 2\theta - r\right) \mathrm{d}r\,\mathrm{d}\theta = 2\left(\int_0^{2\pi} \cos 2\theta\,\mathrm{d}\theta\right)\left(\int_0^1 r^5\,\mathrm{d}r\right) - 2\int_0^{2\pi}\int_0^1 r\,\mathrm{d}r = -2\pi$$

Alternatively, we can use the disk atop the paraboloid as the surface over which to integrate since it shares the boundary with the paraboloid. Projecting onto the xy–plane, we have $\mathbf{p} = \mathbf{k}$, with $\mathcal{R}$, the integration region, the unit disk, and

$$g(x, y, z) = z \implies \nabla g = \langle 0, 0, 1 \rangle \implies |\nabla g \cdot \mathbf{p}| = 1$$

For proper orientation we use $-\nabla g$ for the normal to the surface giving

$$(\nabla \times \mathbf{F}) \cdot \frac{-\nabla g}{|\nabla g \cdot \mathbf{p}|} = \langle xz, -yz, 2 \rangle \cdot \langle 0, 0, -1 \rangle = -2$$

and

$$\text{Circulation} = \iint\limits_{x^2+y^2 \le 1} -2\,\mathrm{d}A = -2 \iint\limits_{x^2+y^2 \le 1} \mathrm{d}A = -2\pi$$

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