INSTRUCTIONS: Books, notes, crib sheets, and electronic devices are not permitted. Write your (1) name, (2) instructor's name, (3) recitation number. Work all problems. Show and explain your work clearly. Note that a correct answer with incorrect or no meaningful supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

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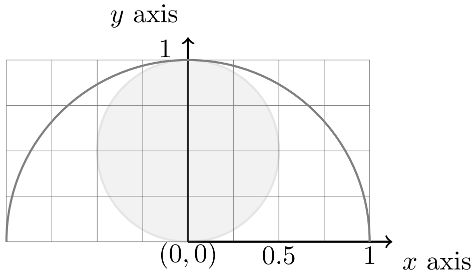
1. (25 points) Consider the upper half of the sphere described in Cartesian coordinates by $x^2 + y^2 + z^2 \leq 1$. Now, remove the volume of the smaller sphere described in spherical coordinates by $\rho = \cos\phi$.

   (a) Make a clear sketch of the geometry of the situation as viewed in a plane of constant θ in the cylindrical coordinate system.
   SOLUTION: The material removed is shaded gray and is in the first two quadrants, as shown below.



   (b) Set up the integral, in cylindrical coordinates, to determine the volume of material remaining.
   SOLUTION: The remaining volume is $V = \int_{\theta=0}^{2\pi} \int_{z=0}^{1} \int_{r=\sqrt{1/4-(z-0.5)^2}}^{\sqrt{1-z^2}} r\, dr\, dz\, d\theta$.

   (c) Set up the integral, in spherical coordinates, to determine the volume of material remaining.
   SOLUTION: The remaining volume is $V = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=\cos\phi}^{1} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$, or
   $V = \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \int_{\phi=\cos^{-1}\rho}^{\pi/2} \rho^2 \sin(\phi)\, d\phi\, d\rho\, d\theta$.

   (d) Evaluate your integral from either part (b) or (c) above to determine the remaining volume.
   SOLUTION: $V = \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \int_{\phi=\cos^{-1}\rho}^{\pi/2} \rho^2 \sin(\phi)\, d\phi\, d\rho\, d\theta$. After the ϕ integration, the integral reduces to
   $V = \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \rho^3\, d\rho\, d\theta$. After the ρ integration we are left with $V = \int_{\theta=0}^{2\pi} \left(\frac{1}{4}\right) d\theta = \frac{\pi}{2}$.

2. (25 points) Use the transformation $u = x - y$ and $v = 2x + y$ to evaluate the integral

$$I = \iint_R (2x^2 - xy - y^2)\, dx\, dy$$

   over the finite region R bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$.

   (a) Solve for x and y in terms of u and v using the given substitution. Be sure to check this because the rest of the problem depends on this result! Check this again.
   SOLUTION: $x = \frac{1}{3}(u + v)$ and $y = \frac{1}{3}(v - 2u)$.

   (b) Transform the original region R_{xy} into its corresponding region R_{uv} in the uv–plane. Make a clear sketch of the new region of integration R_{uv} in the uv–plane. Be sure to label all axes, boundaries, intersection points, etc. on your sketch.
   SOLUTION: The boundaries transform into the rectangular region $4 \leq v \leq 7$ and $-1 \leq u \leq 2$.

   (c) Rewrite the integral for I over the region R_{uv} in the uv–plane in terms of u and v.
   SOLUTION: The Jacobian is $x_u y_v - y_u x_v = 1/3$. Hence the transformed integral becomes

$$\int_{v=4}^{7} \int_{u=-1}^{2} uv \left|\frac{1}{3}\right| du\, dv\,.$$

(d) Evaluate I in terms of u and v.
SOLUTION: The integral evaluates to $I = 33/4$.

3. (25 points) The integral

$$V = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{\cos(\phi)} \rho^2 \sin(\phi)\ d\rho\, d\phi\, d\theta$$

calculates the volume of a spherical object.

(a) Make a clear sketch of the cross-section of the object in a constant θ plane. (This could be a constant θ plane in either cylindrical coordinates or spherical coordinates.) Clearly label the bounding surfaces of the region of integration.
SOLUTION: This is the smaller sphere from problem (1) so see the figure from problem (1).

(b) Express V in spherical coordinates using the order $d\phi\, d\rho\, d\theta$.
SOLUTION:

$$V = \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \int_{\phi=0}^{\cos^{-1}(\rho)} \rho^2 \sin(\phi)\ d\phi\, d\rho\, d\theta$$

(c) Express V in cylindrical coordinates using the order $dr\, dz\, d\theta$.
SOLUTION:

$$V = \int_{\theta=0}^{2\pi} \int_{z=0}^{1} \int_{r=0}^{\sqrt{z-z^2}} \rho^2 \sin(\phi)\ dr\, dz\, d\theta$$

(d) Evaluate any of the integrals above (including the original) to determine the volume V.
SOLUTION: Let's evaluate the original integral. After the ρ integration we are left with

$$V = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \cos^3(\phi) \sin(\phi)\ d\phi\, d\theta$$

Now, evaluate the ϕ integral to arrive at

$$V = \int_{\theta=0}^{2\pi} \frac{1}{12}\ d\theta = \frac{\pi}{6}.$$

4. (25 points) Consider the counterclockwise circular path in the x-y plane around the origin with a radius of R. Also consider the vector function given by $\mathbf{F} = xy\,\mathbf{i} + xy\,\mathbf{j}$.

(a) Sketch the entire path in the x-y plane, including the direction of motion. (This should be free points.)

(b) Clearly give a parametrization for the path, $\mathbf{r}(t)$, including the limits on t. (Again, this should be free points.)
SOLUTION: $\mathbf{r}(t) = R\cos(t)\,\mathbf{i} + R\sin(t)\,\mathbf{j}$ for $0 \le t \le 2\pi$. From this we can calculate the velocity as $\mathbf{v}(t) = -R\sin(t)\,\mathbf{i} + R\cos(t)\,\mathbf{j}$ and the speed as $|\mathbf{v}| = R$. Also, $\tilde{\mathbf{v}}(t) = \mathbf{v} \times \mathbf{k} = R\cos(t)\,\mathbf{i} + R\sin(t)\,\mathbf{j}$

(c) Determine the unit tangent to the path, $\mathbf{T}$.
SOLUTION: $\mathbf{T} = \mathbf{v}/|\mathbf{v}| = -\sin(t)\,\mathbf{i} + \cos(t)\,\mathbf{j}$

(d) Determine the **outward** unit normal to the path, $\mathbf{n}$.
SOLUTION: $\mathbf{n} = \mathbf{T} \times \mathbf{k} = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j}$.

(e) Calculate the **flow** along the path C.
SOLUTION:
$$\text{Flow} = \int \mathbf{F}\cdot\mathbf{T}\,ds = \int_{t=0}^{2\pi} \mathbf{F}\cdot\mathbf{v}\,dt = \int_{t=0}^{2\pi} R^3\cos(t)\sin^2(t) + R^3\cos^2(t)\sin(t)\,dt = 0.$$

(f) Calculate the **flux** along the path C.
SOLUTION:
$$\text{Flux} = \int \mathbf{F}\cdot\mathbf{n}\,ds = \int_{t=0}^{2\pi} \mathbf{F}\cdot\tilde{\mathbf{v}}\,dt = \int_{t=0}^{2\pi} R^3\cos^2(t)\sin(t) + R^3\cos(t)\sin^2(t)\,dt = 0.$$

Projections and distances $\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A}$ $\quad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$ $\quad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

The Second Derivative Test

Suppose $f(x,y)$ and its first and second partial derivatives are continuous in a disk centered at (a,b) and $f_x(a,b) = f_y(a,b) = 0$. Let $D = f_{xx}f_{yy} - f_{xy}^2$.

1. If $D > 0$ and $f_{xx} < 0$ at (a,b), then f has a local maximum at (a,b).
2. If $D > 0$ and $f_{xx} > 0$ at (a,b), then f has a local minimum at (a,b).
3. If $D < 0$ at (a,b), then f has a saddle point at (a,b).
4. If $D = 0$ at (a,b), then the test is inconclusive.

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$\begin{aligned} f(x,y) \quad = \quad & f(x_0,y_0) + \Big[(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\Big] \\ & + \frac{1}{2!}\Big[(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\Big] \\ & + \frac{1}{3!}\Big[(x-x_0)^3 f_{xxx}(x_0,y_0) + 3(x-x_0)^2(y-y_0)f_{xxy}(x_0,y_0) \\ & \qquad + 3(x-x_0)(y-y_0)^2 f_{xyy}(x_0,y_0) + (y-y_0)^3 f_{yyy}(x_0,y_0)\Big] + \cdots \end{aligned}$$

Linear approximation error

$$|E(x,y)| \le \frac{1}{2}M(|x-x_0| + |y-y_0|)^2, \qquad \text{where } \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \le M$$

Polar coordinates $x = r\cos\theta \quad y = r\sin\theta \quad r^2 = x^2 + y^2 \quad dA = dx\,dy = r\,dr\,d\theta$

Cylindrical and spherical coordinates

Cylindrical to Rectangular	Spherical to Cylindrical	Spherical to Rectangular
$x = r\cos\theta$	$r = \rho\sin\phi$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$z = \rho\cos\phi$	$y = \rho\sin\phi\sin\theta$
$z = z$	$\theta = \theta$	$z = \rho\cos\phi$

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Substitutions in multiple integrals

$$\int\!\!\int_R f(x,y)\,dx\,dy = \int\!\!\int_S f\left(x(u,v), y(u,v)\right)\,|J(u,v)|\,du\,dv \qquad \text{where} \qquad J(u,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Mass, moments, and center of mass Mass $M = \int\!\!\int_R \delta\,dA$

$$\text{Moments} \quad M_x = \int\!\!\int_R y\,\delta\,dA \qquad M_y = \int\!\!\int_R x\,\delta\,dA \qquad \text{Center of mass} \quad \bar{x} = M_y/M \qquad \bar{y} = M_x/M$$

Flow and flux Flow $= \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot\mathbf{V}\,dt = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy$

$$\text{Flux} = \int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C M\,dy - N\,dx \quad \text{where} \quad \mathbf{n} = \mathbf{T}\times\mathbf{k}$$