1. (25 points) Consider the upper half of the sphere described in Cartesian coordinates by \( x^2 + y^2 + z^2 \leq 1 \). Now, remove the volume of the smaller sphere described in spherical coordinates by \( \rho = \cos \phi \).

(a) Make a clear sketch of the geometry of the situation as viewed in a plane of constant \( \theta \) in the cylindrical coordinate system.
(b) Set up the integral, in cylindrical coordinates, to determine the volume of material remaining.
(c) Set up the integral, in spherical coordinates, to determine the volume of material remaining.
(d) Evaluate your integral from either part (b) or (c) above to determine the remaining volume.

2. (25 points) Use the transformation \( u = x - y \) and \( v = 2x + y \) to evaluate the integral \( I = \int \int_R (2x^2 - xy - y^2) \, dx \, dy \) over the finite region \( R \) bounded by the lines \( y = -2x + 4 \), \( y = -2x + 7 \), \( y = x - 2 \), and \( y = x + 1 \).

(a) Solve for \( x \) and \( y \) in terms of \( u \) and \( v \) using the given substitution. Be sure to check this because the rest of the problem depends on this result! Check this again.
(b) Transform the original region \( R_{xy} \) into its corresponding region \( R_{uv} \) in the \( uv \)–plane. Make a clear sketch of the new region of integration \( R_{uv} \) in the \( uv \)–plane. Be sure to label all axes, boundaries, intersection points, etc. on your sketch.
(c) Rewrite the integral for \( I \) over the region \( R_{uv} \) in the \( uv \)–plane in terms of \( u \) and \( v \).
(d) Evaluate \( I \) in terms of \( u \) and \( v \).

3. (25 points) The integral \( V = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{\cos(\phi)} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta \) calculates the volume of a spherical object.

(a) Make a clear sketch of the cross-section of the object in a constant \( \theta \) plane. (This could be a constant \( \theta \) plane in either cylindrical coordinates or spherical coordinates.) Clearly label the bounding surfaces of the region of integration.
(b) Express \( V \) in spherical coordinates using the order \( d\phi \, d\rho \, d\theta \).
(c) Express \( V \) in cylindrical coordinates using the order \( dr \, dz \, d\theta \).
(d) Evaluate any of the integrals above (including the original) to determine the volume \( V \).

4. (25 points) Consider the counterclockwise circular path in the \( x\)-\( y \) plane around the origin with a radius of \( R \). Also consider the vector function given by \( \mathbf{F} = xy \mathbf{i} + xy \mathbf{j} \).

(a) Sketch the entire path in the \( x\)-\( y \) plane, including the direction of motion. (This should be free points.)
(b) Clearly give a parametrization for the path, \( \mathbf{r}(t) \), including the limits on \( t \). (Again, this should be free points.)
(c) Determine the unit tangent to the path, \( \mathbf{T} \).
(d) Determine the outward unit normal to the path, \( \mathbf{n} \).
(e) Calculate the flow along the path \( C \).
(f) Calculate the flux along the path \( C \).

OVER
Projections and distances  
\[ \text{proj}_A \mathbf{B} = \left( \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \right) \mathbf{A} \quad d = \frac{\lvert \mathbf{F} \times \mathbf{n} \rvert}{\lvert \mathbf{n} \rvert} \]

Arc length, frenet formulas, and tangential and normal acceleration components  
\[ ds = \lvert \mathbf{v} \rvert \, dt \quad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{\lvert \mathbf{v} \rvert} \quad \mathbf{N} = \frac{d\mathbf{T}/ds}{d\mathbf{T}/ds} = \frac{d\mathbf{T}/dt}{d\mathbf{T}/ds} \quad \mathbf{B} = \mathbf{T} \times \mathbf{N} \]

\[ \frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \quad \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N} \]

\[ \kappa = \frac{\left| \mathbf{v} \times \mathbf{a} \right|}{\lvert \mathbf{v} \rvert^3} = \frac{\left| f''(x) \right|}{1 + (f'(x))^2}^{3/2} - \frac{\lvert \dot{x}y - \dot{y}x \rvert}{\lvert x^2 + y^2 \rvert^{3/2}} \]

\[ \tau = \frac{d\mathbf{B}}{ds} \]

The Second Derivative Test  
Suppose \( f(x, y) \) and its first and second partial derivatives are continuous in a disk centered at \((a, b)\) and \( f_a(a, b) = f_y(a, b) = 0 \). Let \( D = f_{xx}f_{yy} - f_{xy}^2 \).

1. If \( D > 0 \) and \( f_{xx} < 0 \) at \((a, b)\), then \( f \) has a local maximum at \((a, b)\).
2. If \( D > 0 \) and \( f_{xx} > 0 \) at \((a, b)\), then \( f \) has a local minimum at \((a, b)\).
3. If \( D < 0 \) at \((a, b)\), then \( f \) has a saddle point at \((a, b)\).
4. If \( D = 0 \) at \((a, b)\), then the test is inconclusive.

Directional derivative, discriminant, and Lagrange multipliers  
\[ \frac{df}{ds} = (\nabla f) \cdot \mathbf{u} \quad f_{xx}f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda \nabla g, \quad g = 0 \]

Taylor's formula (at the point \((x_0, y_0)\))  
\[ f(x, y) = f(x_0, y_0) + \left[ (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) \right] + \frac{1}{2!} \left[ (x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0)f_{xy}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0) \right] + \cdots \]

Linear approximation error  
\[ |E(x, y)| \leq \frac{1}{2} M \lvert x - x_0 \rvert \lvert y - y_0 \rvert^2, \quad \text{where max}\{\lvert f_{xx} \rvert, \lvert f_{xy} \rvert, \lvert f_{yy} \rvert \} \leq M \]

Polar coordinates  
\[ x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad dA = dx \, dy = r \, dr \, d\theta \]

Cylindrical and spherical coordinates  

<table>
<thead>
<tr>
<th>Cylindrical to Rectangular</th>
<th>Spherical to Cylindrical</th>
<th>Spherical to Rectangular</th>
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</thead>
<tbody>
<tr>
<td>( x = r \cos \theta )</td>
<td>( r = \rho \sin \phi )</td>
<td>( x = \rho \sin \phi \cos \theta )</td>
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<td>( y = r \sin \theta )</td>
<td>( z = \rho \cos \phi )</td>
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<td>( z = z )</td>
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\[ dV = dx \, dy \, dz = r \, dr \, d\theta \, dz = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

Substitutions in multiple integrals  
\[ \int \int_R f(x, y) \, dx \, dy = \int \int_S f(x(u, v), y(u, v)) \, |J(u, v)| \, du \, dv \]  
where \( J(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \)

Mass, moments, and center of mass  
\[ \text{Mass} \quad M = \int \int_R dA \]

\[ \text{Moments} \quad M_x = \int \int_R y \, dA \quad M_y = \int \int_R x \, dA \quad \text{Center of mass} \quad \bar{x} = M_y/M \quad \bar{y} = M_x/M \]

Flow and flux  
\[ \text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot \mathbf{v} \, dt \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M \, dx + N \, dy \]

\[ \text{Flux} = \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C M \, dy - N \, dx \]  
where \( \mathbf{n} = \mathbf{T} \times \mathbf{k} \)